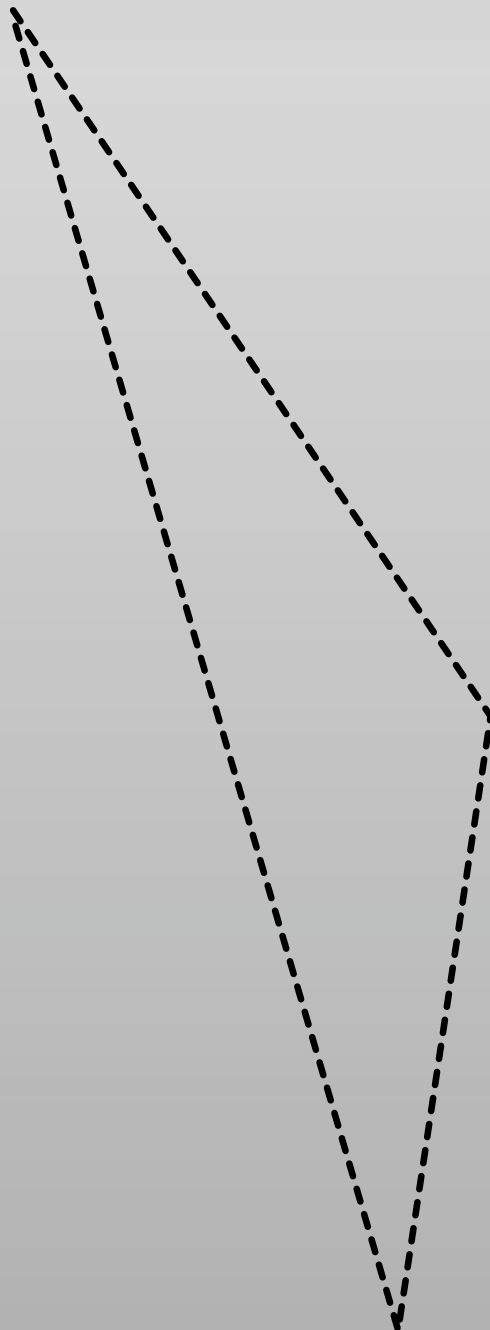


Liquidity Risk meets Economic Capital and RAROC

A framework for measuring liquidity risk in banks

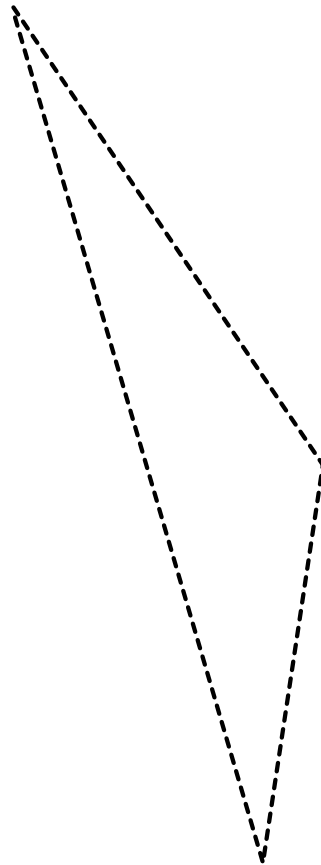
Kolja Loebnitz



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LIQUIDITY RISK MEETS ECONOMIC CAPITAL
AND RAROC:
A FRAMEWORK FOR MEASURING LIQUIDITY
RISK IN BANKS

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Abstract

While banks and regulators use sophisticated mathematical methods to measure a bank's solvency risk, they use relatively simple tools for a bank's liquidity risk such as coverage ratios, sensitivity analyses, and scenario analyses. In this thesis we present a more rigorous framework that allows us to measure a bank's liquidity risk within the standard economic capital and RAROC setting.

In the first part, we introduce the concept of liquidity cost profiles as a quantification of a bank's illiquidity at balance sheet level. The profile relies on a nonlinear liquidity cost term that formalizes the idea that banks can run up significant value losses, or even default, when their unsecured borrowing capacity is severely limited and they are required to generate cash on short notice from its asset portfolio in illiquid secondary asset markets. The liquidity cost profiles lead to the key concept of liquidity-adjusted risk measures defined on the vector space of balance sheet positions under liquidity call functions. We study the model-free effects of adding, scaling, and mixing balance sheets. In particular, we show that convexity and positive super-homogeneity of risk measures is preserved in terms of positions under the liquidity adjustment, given certain moderate conditions are met, while coherence is not, reflecting the common idea that size does matter in the face of liquidity risk. Nevertheless, we argue that coherence remains a natural assumption at the level of underlying risk measures for its reasonable properties in the absence of liquidity risk. Convexity shows that even under liquidity risk the concept of risk diversification survives. In addition, we show that in the presence of liquidity risk a merger can create extra risk. We conclude the first part by showing that a liquidity-adjustment of the well-known Euler capital allocation principle is possible without losing the soundness property that justifies the principle. However, it is in general not possible to combine soundness with the total allocation property for both the numerator and the denominator in liquidity-adjusted RAROC.

In the second part, we present an illustration of the framework in the context of a semi-realistic economic capital setting. We characterize the bank's funding risk with the help of a Bernoulli mixture model, using the bank's capital losses as the mixing variable, and use standard marginal risk models for credit, market, and operational risk. After formulate the joint model using a copula, we analyze the impact of balance sheet composition on liquidity risk. Furthermore, we derive a simple, robust, and efficient numerical algorithm for the computation of the optimal liquidity costs per scenario.

Liquidity-adjusted risk measures could be a useful addition to banking regulation and bank management as they capture essential features of a bank's liquidity risk, can be combined with existing risk management systems, possess reasonable properties under portfolio manipulations, and lead to an intuitive risk ranking of banks.

Samenvatting

Banken en toezichthouders gebruiken geavanceerde wiskundige methoden voor het bepalen van het risico van een bank met betrekking tot solvabiliteit, maar ze gebruiken relatief eenvoudige methoden voor het liquiditeitsrisico van een bank, zoals dekkinggraden, gevoeligheidsanalyses, en scenario-analyses. In dit proefschrift presenteren we een meer structurele aanpak die ons in staat stelt het liquiditeitsrisico van een bank te meten binnen het gebruikelijke kader van ‘Economic Capital’ en RAROC.

In het eerste gedeelte introduceren we het begrip ‘liquiditeitskosten-profiel’ als weergave van de mate van illiquiditeit van een bank op balansniveau. Dit begrip berust op een niet-lineaire term voor liquiditeitskosten, die voortkomt uit het verschijnsel dat banken aanzienlijke verliezen kunnen oplopen, en zelf failliet kunnen gaan, wanneer hun mogelijkheid om ongedekte leningen aan te gaan sterk beperkt is, en ze gedwongen zijn op korte termijn cash te genereren uit hun portefeuille van activa op een illiquide financiële markt. Liquiditeitskosten-profielen leiden tot het sleutelbegrip ‘liquiditeits-aangepaste risicomaten’, gedefinieerd op de vectorruimte van balansposities onderhevig aan plotselinge vraag naar liquiditeit (‘liquidity calls’). We bestuderen effecten van het samenvoegen, schalen, en combineren van balansen. In het bijzonder laten we zien dat de eigenschappen van convexiteit en positief-superhomogeniteit van risicomaten behouden blijft, onder redelijk ruime aannamen, terwijl dat niet geldt voor de eigenschap van coherentie. Dit weerspiegelt het feit dat omvang er wel degelijk toe doet als het om liquiditeit gaat, maar we betogen dat desondanks coherentie wel een natuurlijke aanname blijft op het niveau van onderliggende risicomaten. De eigenschap van convexiteit geeft aan dat zelfs onder liquiditeitsrisico het begrip risico-diversificatie van toepassing blijft. Daarnaast laten we zien dat in aanwezigheid van liquiditeitsrisico, het samenvoegen van balansen (een ‘merger’) extra risico kan creëren. We sluiten het eerste gedeelte af met een stuk waarin we laten zien dat de aanpassing voor liquiditeit van het welbekende Euler-allocatie principe mogelijk is, met inachtneming van het begrip ‘soundness’ dat dit principe rechtvaardigt. Echter, het is in het algemeen niet haalbaar dit begrip te combineren met volledige allocatie van zowel de teller als de noemer in RAROC, aangepast voor liquiditeit.

In het tweede gedeelte illustreren we de aanpak aan de hand van een semi-realistische model voor economisch kapitaal. We karakteriseren het financieringsrisico met behulp van een ‘Bernoulli mixing’ model, waarbij we de kapitaalsverliezen van een bank als ‘mixing’variabele nemen, en standaardmodellen gebruiken voor het krediet-, markt- en operationeel risico. Nadat we een model voor de gezamenlijke verdeling hebben geformuleerd in termen van zogenaamde copula’s, analyseren we de impact van de samenstelling van de balans op liquiditeitsrisico. Daarnaast leiden we een eenvoudig,

robuust en efficiënt algoritme af voor de berekening van optimale liquiditeitskosten per scenario.

Liquiditeits-aangepaste risicomaten kunnen een bruikbare aanvulling leveren op het reguleren en besturen van banken, omdat ze essentiële aspecten van het liquiditeitsrisico van een bank weergeven, ze gecombineerd kunnen worden met bestaande systemen voor risicomanagement, ze aannemelijke eigenschappen hebben onder aanpassingen van portefeuilles, en leiden tot een intuïtieve rangschikking van banken.

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Public working papers

Loebnitz, K. and Roorda, B. (2011). Liquidity risk meets economic capital and raroc.
<http://ssrn.com/abstract=1853233>.

1

Introduction

1.1 Problem statement and research questions

Bank managers have an incentive to manage their business prudently so as to maximize economic value while avoiding the occurrence of default.¹ Default can occur through two main mechanisms: (1) Technical insolvency: the asset portfolio value drops below the notional value of the liability portfolio and (2) illiquidity: the bank is unable to pay its monetary obligations on time, despite being technically solvent.² Since it is in general not possible to earn profits in financial markets without being exposed to risk, i.e., there is no “free lunch”, banks actively take on risks. For the bank these risks involve, value losses on its trading positions due to price fluctuations, i.e., market risk, losses on its portfolio of loans and bonds, i.e., credit risk, and losses related to inadequate or failed internal processes, fraud and litigation, i.e., operational risk. As these value losses decrease the bank’s capital position and hence endanger the bank’s continuity, the management of these risks is paramount. Nowadays banks use partly due to the introduction of quantitative requirements by the supervisory authorities sophisticated mathematical methods to measure and manage the technical insolvency leg of its own default risk. In particular, banks measure their solvency risk with the help of probability theory, the theory of stochastic processes, statistics, and the theory of monetary risk measures after Artzner et al. (1999). In addition, banks employ a wide range of quantitative tools to manage their risk exposure, such as diversification,

¹Our ideas apply to any financial investor or even economic agent, but we emphasize the situation of banks due some particularities of their business model with regard to liquidity risk.

²From an “external” market perspective the two channels are usually subsumed under the credit risk of the bank (probability of default). However, from an “internal” bank perspective it is meaningful to distinguish between the two mechanisms. We take the latter position in this thesis.

hedging, insuring, and securization.

In contrast to solvency risk, liquidity risk of banks is mostly assessed by relatively ad-hoc means, typically involving a combination of financial ratios, constraints, sensitivity analyses, and stress scenario analyses. Recently, the Basel Committee proposed, under the header of Basel III, an attempt to harmonize liquidity risk supervision (BIS, 2010). In particular, the committee suggests that banks must show that they (1) can survive for 30 days under an acute liquidity stress scenario specified by the supervisors and (2) have an acceptable amount of stable funding based on the liquidity characteristics of the bank's on- and off-balance sheet assets and activities over a one year horizon.

While we think that having standardized liquidity risk regulations is a step in the right direction, we also note that the level of sophistication of these new regulations is comparable to the first Basel Accord for solvency risk³ and is similar to what is already common practice in most large banks. We believe that liquidity risk measurement and management would benefit from a formal treatment akin to solvency risk. While we do not claim that a mathematical treatment of liquidity risk necessarily leads to better liquidity risk management in practice, we believe it has an advantage over an ad-hoc approach in that it allow us to study its non-trivial properties under various assumptions. This way we can illustrate its benefits as well as its limitations in a consistent manner.⁴ In contrast, the use of mathematical models in the financial world has been criticized by some as being partially responsible for the recent Subprime crisis. While we agree that sometimes the use of mathematical models can lead to real life problems, we argue that models themselves are not the problem, only the inappropriate use of them by people.⁵ Furthermore, in this thesis we do not focus on a particular class of restrictive probability models but a general and flexible mathematical framework. A more practical argument for the need of a mathematical treatment of liquidity risk is one of proportionality. The series of bank defaults due to illiquidity, such as Bear Stearns, Lehman Brothers, and Northern Rock, showed that liquidity risk as a default channel is at least as important as default by insolvency. Consequently, if regulators and bank managers believe in the usefulness of a formal treatment of solvency risk in banks, then they should also support it for liquidity risk.

Even though most people think of stochastic models when they think of mathematical modeling, we often need a mathematical framework first that clarifies what actually should be modeled before such models can be developed. Examples of formalisms are the capital adequacy framework after Artzner et al. (1999) for a bank's solvency risk and Cramér-Lundberg's ruin framework in actuarial mathematics (see, e.g., Buehlmann

³This is not surprising, considering the time and effort it took to advance the solvency regulations from Basel I to the level of Basel II.

⁴Perhaps a good example for the value of a mathematical approach in the context of financial risk measurement is that Artzner et al. (1999) show that VaR is in general not subadditive and hence using it can lead to some unpleasant results under certain assumptions.

⁵However, we think there is a resemblance to the topic of gun control and the argument that guns do not kill, but people do. While this may be correct in a sense, any sincere policy maker has to take into account what the combination of impulsiveness of people's actions and the availability of guns can lead to. The same may be said about the availability of mathematical models.

(1997)). Admittedly, it sometimes is straightforward what needs to be modeled and no elaborate discussion is needed. However, we think that this does not apply in this case and liquidity risk would benefit from the development of a sound mathematical framework. The main goal of this thesis is to develop such a formalism.

During the recent financial crisis, we have witnessed that banks that were technically solvent and acceptable in the capital adequacy sense experienced severe difficulties to stay liquid and some even failed because of illiquidity (Morris and Shin, 2009). Extending this phenomenon a bit, we believe it is reasonable to say that:

Key observation 1: In practice, adequate capitalization in terms of economic capital (EC)⁶ is not a sufficient condition for adequate liquidity of a bank.

There are two ways to look at this observation. On the one hand, this should not surprise us because conceptually liquidity risk does not enter the standard capital adequacy framework. Of course, bank managers, regulators, and rating agencies are aware of that as they assess a bank's liquidity risk typically with the help of a cash flow mismatch framework and various stress scenarios. This analysis is usually completely divorced from the capital adequacy framework. On the other hand, we would in principle expect that a bank for which its available capital is higher than its EC, provided that the latter includes all material risks, will not suffer from funding problems. Consequently, during the recent Subprime crisis investors must have doubted the comprehensiveness and/or some of the assumptions of the bank's EC models and hence believed that banks underestimated their EC and/or overstated their available capital.

This brings us to a second important observation. Brunnermeier et al. (2009) maintain that linking capital adequacy and liquidity risk is crucial to strengthen the resilience of the financial system as a whole.

Key observation 2: "Financial institutions who hold assets with low market liquidity and long-maturity and fund them with short-maturity assets should incur a higher capital charge. We believe this will internalise the systemic risks these mismatches cause and incentive banks to reduce funding liquidity risk." (Brunnermeier et al., 2009, p. 37)⁷

The final observation deals with the idea that we ought to look at the riskiness of the asset and its funding structure.

Key observation 3: "Conceptually, regulatory capital [economic capital] should be set aside against the riskiness of the combination of an asset and its funding, since the riskiness of an asset [portfolio] depends to a large extent on the way it is funded." (Brunnermeier et al., 2009, p. 41).

⁶In this thesis, we use the term EC instead of regulatory capital (RC), because we would like to abstract from the exact functional form of the actual RC after Basel II. However, the ideas presented in this thesis may also be useful for future RC formulations.

⁷To avoid confusion, the mismatch cannot refer to the mismatches related to interest rate risk in the banking book of the bank, because these effects are typically already included in the EC.

We think that because capital requirements and economic capital play such a prominent role as a management control tool within banks and as a signaling tool in the financial world, it would be advantageous to be able to adjust the standard capital adequacy framework for some notion of liquidity risk. However, bringing solvency and liquidity risk measurement of banks together in one conceptual framework is not done in practice (cf., Basel III, BIS (2010)), mostly because it is believed that the two concepts do not mix well. One reason is that there is the belief that stochastic modeling of liquidity risk is particular hard and hence difficult to combine with solvency risk modeling.⁸ Another possible explanation is that the capital adequacy setting is typically static which is reasonable for solvency risk but problematic for liquidity risk because of the importance of timing of cash flows and other dynamic elements. Conversely, EC should conceptually take into account all potential risks that can lead to a decline of a bank's capital and this includes, e.g., values losses from distress liquidation in times of liquidity problems (see also Klaassen and van Eeghen (2009) on p. 44). While we agree that bringing the two concepts together in one framework is not necessarily compelling, we believe that it is pragmatic and the theoretical nature of a liquidity risk adjustment for EC is the fundamental concern of this thesis. We would like to stress the fact that this thesis is about laying the theoretical groundwork for a more rigorous approach to liquidity risk measurement and not about concrete applications.

1.2 Liquidity risk and banks

What is the liquidity risk of a bank? There are two basic dimensions that can be associated with liquidity risk: (1) costs in the widest sense arising from difficulties to meet obligations (no default) and more severely (2) the inability of a bank to generate sufficient cash to pay its obligations (default). The former comes in degrees, whereas the latter is a binary (yes or no) concept. We will refer to the cost character as *Type 1* liquidity risk and the default character as *Type 2* liquidity risk from here on. The costs of Type 1 liquidity risk include increased funding costs in capital markets due to systematic or idiosyncratic factors but also value losses due to the liquidation of assets in periods of distress. We do not focus on the former costs in this thesis because they are typically already included as an ALM module in a bank's EC.

In practice banks commonly associate the notion of liquidity risk, sometimes referred to as funding liquidity risk or contingency liquidity risk (Matz and Neu, 2007), with the binary dimension (BIS, 2008a; Nikolaou, 2009), although the cost dimension also plays an important role in the context of interest rate risk and *fund transfer pricing* (FTP). While bank managers, regulators, and credit rating agencies have recognized liquidity risk as an important issue for a long time, it has not received the same attention as solvency risk. Still, the Basel Committee on Banking Supervision has published

⁸For instance, in BIS (2009) on p. 6 we read, "Not all risks can be directly quantified. Material risks that are difficult to quantify in an economic capital framework (eg funding liquidity risk or reputational risk) should be captured in some form of compensating controls (sensitivity analysis, stress testing, scenario analysis or similar risk control processes)."

several documents over the years specifically related to the analysis and management of liquidity risk in financial institutions (BIS, 2000, 2006, 2008b,a, 2010).

In BIS (2008b) the Basel Committee suggests that banks should (1) analyze the bank's ability to pay their obligations via a forward looking cumulative cash flow mismatch framework under a small finite number of stress scenarios and (2) maintain a document that explains who does what when severe liquidity problems arise, called a *contingency funding plan*, without, however, prescribing anything specific. This changed with the release of Basel III (BIS, 2010). In their latest document the Basel Committee complements their previous catalog of best practices with two regulatory standards for liquidity risk: (1) the Liquidity Coverage Ratio (LCR) and (2) the Net Stable Funding Ratio (NSFR). The LCR amounts to dividing the value of the stock of unencumbered high-quality liquid assets of the bank in stressed conditions by the total net cash outflows over the next 30 calendar days under a prescribed stress scenario. This ratio should be greater than one at all times. It is hoped that this requirement promotes short-term resilience by ensuring that a bank has sufficient high-quality liquid assets to survive a significant stress scenario lasting for one month. The NSFR sets the available amount of stable funding in relation to the required amount of stable funding, which should be greater than one at any time. Basel Committee hopes that the NSFR limits the over-reliance on short-term wholesale funding during good times and encourages banks to assess their liquidity risk across all on- and off-balance sheet items better. While we believe that harmonizing liquidity risk regulation is a step in the right direction and that both LCR and NSFR capture essential features of a bank's liquidity risk, there are in our opinion some limitations to this approach as well. Mainly, the level of sophistication is comparable to the risk weighting approach of Basel 1 accord in that it relies heavily on a rather crude classification of assets and liabilities in terms of liquidity risk characteristics. Furthermore, there are no considerations of the possibility of different types of stress scenarios. And finally, due to its deterministic character, banks cannot use the comprehensive stochastic risk modeling they already do for EC to support the liquidity risk analysis, which is unfortunate as we know that most of the time liquidity problems are preceded by solvency problems.

Up until Basel III becomes binding for some banks, the actual regulatory requirements regarding liquidity risk vary from country to country, ranging from quantitative to qualitative measures, as well as a mixture of the two types.⁹ Apart from qualitative measures such as adequate management control structures, quantitative requirements are either based on a stock approach or a maturity mismatch approach, or a combination of the two. The stock approach requires banks to hold a minimum amount of cash or near-cash assets in relation to liabilities, mostly based on heuristic rules. Internally, banks adhere to the BIS best practices of employing a combination of a forward looking cumulative cash flow mismatch framework and stress scenarios to analyze liquidity risk (BIS, 2006; Deutsche Bank, 2008, p. 102; The Goldman Sachs Group, 2006, p. 97;

⁹For instance, Germany and Austria use mostly quantitative regulations, whereas the United States use qualitative regulations. UK, France and the Netherlands use a mixture (Lannoo and Casey, 2005).

JPMorgan Chase & Co., 2007, p. 70; Citigroup Inc., 2008, p. 102; UBS AG, 2008, p. 151). Likewise, credit rating agencies, factor the results of similar liquidity risk analyses into the credit ratings they give to banks (Standard&Poor's, 2004; Martin, 2007, p. 6). Summing up, practitioners deem their actual portfolio position acceptable in terms of liquidity risk as long as it meets a number of constraints expressed in terms of coverage ratios (e.g., cash capital ratio), limits (e.g., principal amount of debt maturing in any particular time interval), and stress scenario analysis outcomes in a cash flow mismatch setting (e.g., entities should be self-funded or net providers of "liquidity" under each stress scenario). Violations of any constraints lead to corrective actions. In addition, national central banks monitor the adequacy of these analyses, completing the circle similar to capital adequacy regulations. With the introduction of Basel III, liquidity risk regulation will indeed be harmonized but it seems that most large banks are already using something close to LCR and NSFR.

We can conclude that, while the concept of liquidity acceptability used in practice and under the new regulations are not as elegant and rigorous as the formal concept of *acceptability* proposed by modern risk measure theory following Artzner et al. (1999) and the formalism presented in this thesis, they fulfill the same purpose and are not necessarily less valuable. For this reason, we believe that the framework presented in this thesis should be seen as a useful addition to the decision support toolbox of bank managers and financial regulators and not as a replacement of existing liquidity risk management tools.

1.3 Research approach and outline of contributions

The main concern of this thesis is to make economic capital and risk-adjusted return on capital (RAROC) sensitive to Type 1 and Type 2 liquidity risk of a bank without distorting the character and purpose of these tools. This requires the development of a fundamental liquidity risk formalism that is flexible enough to be applied to any form of bank, much like the economic capital and RAROC formalism. For this purpose, we introduce in Chapter 2 the concept of a liquidity cost profile as a quantification of a bank's illiquidity at balance sheet level. The profile relies on a nonlinear liquidity cost term that takes into account both the bank's exposure to funding liquidity risk and market liquidity risk. The cost term formalizes the idea that banks can run up significant value losses, or even default, when their unsecured borrowing capacity is severely limited and they are required to generate cash on short notice from its asset portfolio in illiquid secondary asset markets. The reasoning behind the liquidity cost term and our formalism is that idiosyncratic funding problems of a bank can potentially be caused by asymmetric information between banks and fund providers. In such situations fund providers have doubts about the bank's creditworthiness and before the bank can remove the uncertainty regarding their situation, funding is problematic for the bank. During such times the bank needs a sufficiently large asset liquidity reserve, i.e., a portfolio of unencumbered liquid assets, to service its debt obligations and buy

enough time to improve its reputation. However, due to limited market liquidity during such times any distress sales would lead to value losses that decrease the bank's capital (Type 1 liquidity risk) or even worse the bank could default because it cannot generate enough cash from its asset position (Type 2 liquidity risk). The reasoning behind our formalism is similar to the idea behind LCR and liquidity calls in our formalism are closely related to the short-term total net cash flow in stress periods used in Basel III.

Mathematically, we start in Chapter 2 with a simple timeless and deterministic setting. We begin by introducing the concept of a bank's asset portfolios and by assuming that the proceeds of liquidating a portion of the bank's asset portfolio is concave and bounded from above by the frictionless linear Mark-to-Market (MtM) value of the assets (see Definition 2.4). In addition, we postulate that any liquidity call (cash obligation) a bank needs to generate in a distress situation is generated by the bank so that the liquidity costs, i.e., the difference between the MtM value and the actual liquidation value (see Definition 2.4), are minimized. That means that the liquidity costs term is the result of a nonlinear, but fortunately convex constrained optimization problem (see Definition 2.8). After characterizing the optimization problem in Lemma 2.10, the liquidity cost profile of a bank is defined as the unique function mapping for a given asset portfolio each non-negative liquidity call to the portfolio's marginal liquidity costs (see Definition 2.11). Integrating this function from zero to the liquidity call gives the optimal liquidity costs.

Equipped with these tools, we turn towards the standard two period risk measurement setting of economic capital and financial risk measure theory after Artzner et al. (1999). We introduce the concept of asset and liability pairs (balance sheets) and liquidity call functions (see Definition 2.20). The latter maps portfolio pairs to random nonnegative liquidity calls and is used to represent the funding liquidity risk of banks. The notions of random liquidity calls, random proceeds, and hence random optimal liquidity costs, lead to the key concept of liquidity-adjusted risk measures defined on the vector space of asset and liability pairs or balance sheets under liquidity call functions (see Definition 2.25). Next, we study the model-free effects of adding, scaling, and mixing balance sheets which are summarized in Theorem 2.26. In particular, we show that convexity and positive super-homogeneity of risk measures is preserved in terms of positions under the liquidity adjustment, given certain moderate conditions are met, while coherence is not, reflecting the common idea that size does matter. We also indicate how liquidity cost profiles can be used to determine whether combining positions is beneficial or harmful. In particular, we show that combining positions with the same marginal liquidity costs generally leads to an increase of total liquidity costs. This effect works in opposite direction of the subadditivity of the underlying risk measure, showing that a merger can create extra risk in the presence of liquidity risk. Afterwards, we address the liquidity-adjustment of the well-known Euler allocation principle for risk capital. We show that such an adjustment is possible without losing the soundness property (see Definition 2.28) that justifies the Euler principle. However, it is in general not possible to combine soundness with the total allocation property for

both the numerator and the denominator in liquidity-adjusted RAROC.

Little academic research has been done on incorporating liquidity risk into economic capital and RAROC. The recent papers by Jarrow and Protter (2005), Ku (2006), Acerbi and Scandolo (2008), and Anderson et al. (2010) are among the few papers that look at the intersection between liquidity risk, capital adequacy, and risk measure theory and hence share similar objectives with our thesis. Common to all four papers is the idea that a part of an asset portfolio must be liquidated in illiquid secondary asset markets and as a result liquidity costs relative to the frictionless MtM are incurred. Risk measures are consequently defined on the portfolio value less the liquidity costs, except for Anderson et al. (2010) who choose a different approach. We follow the line of reasoning of the former papers and we emphasize, similar to Acerbi and Scandolo (2008), that liquidity risk naturally changes the portfolio value from a linear to a non-linear function of the portfolio positions.¹⁰ Despite the similarities with Acerbi and Scandolo (2008) and Anderson et al. (2010), there are important differences between our works. In Acerbi and Scandolo (2008) funding liquidity risk can be interpreted as exogenous. In contrast, we use the concept of asset and liability pairs to internalize funding liquidity risk to some degree with the help of liquidity call functions. The latter maps asset and liability pairs to random liquidity calls that must be met by the bank on short notice by liquidating part of its asset portfolio without being able to rely on unsecured borrowing. This is similar to Anderson et al. (2010)'s short-term cash flow function. By imposing a liquidity call constraint, we can investigate the optimization problem as well as emphasize Type 2 liquidity risk. Of the above papers, we are the only one who stress the effects of Type 2 liquidity risk on concepts such as risk diversification and capital requirements, which turns out to be of importance. In addition, we also discuss the problem of the allocation of liquidity-adjusted economic capital and RAROC to business units, which none of the above papers do. For a more detailed discussion of the related literature we refer the reader to the introduction of Chapter 2.

After introducing the basic liquidity risk formalism and analyzing its properties, we turn in Chapter 3 towards a detailed illustration of the formalism in the context of a semi-realistic economic capital model. The goal of the chapter is threefold: 1.) present a reasonable, albeit stylized, modeling of liquidity risk in conjunction of the typical risk types of a bank, 2.) illustrate what impact the balance sheet composition has on liquidity risk, and 3.) illustrate the relevance of the previously derived formal results. For the second goal, we associate three balance sheet compositions to three different types of banks commonly found in practice: retail banks, universal banks, and investment banks. We characterize the bank's funding risk with the help of a Bernoulli mixture model, using the bank's capital losses as the mixing variable, and use standard marginal risk models for credit, market, and operational risk. We derive the joint model using a copula approach. Furthermore, we introduce a simple, robust, and efficient numerical algorithm based on the results in Lemma 2.10 for the computation

¹⁰In the classical setting the portfolio value is a nonlinear function of risk factors but a linear function of the portfolio positions.

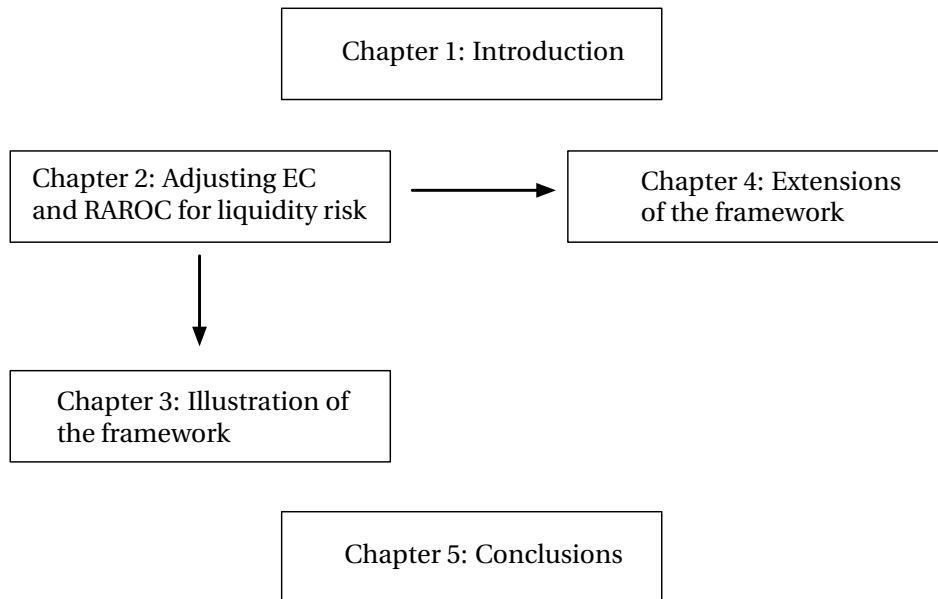


Figure 1.1: Thesis outline.

of the optimal liquidity costs per scenario. While the optimization problem behind the liquidity cost term is convex and hence readily solvable with standard software tools, our algorithm is generally more efficient. We show that even our simple but reasonable implementation of liquidity risk modeling can lead to a significant deterioration of capital requirements and risk-adjusted performance for banks with safe funding but illiquid assets, exemplified by the retail bank, and banks with liquid assets but risky funding, exemplified by the investment bank. In addition, we show that the formal results of Theorem 2.26 are relevant, especially the super-homogeneity result of liquidity-adjusted risk measures. Bank size and the nonlinear scaling effects of liquidity risk become very apparent for banks that have to rely on a large amount of fire selling.

In Chapter 4 we briefly discuss some extensions of the basic liquidity risk formalism, including portfolio dynamics, more complicated proceed functions, and an alternative risk contribution allocation scheme.

1.4 Thesis outline

The thesis is organized as follows (see Figure 1.1): in Chapter 2 we introduce the basic liquidity risk formalism, and derive our main mathematical results. In Chapter 3 we present a detailed illustration of the formalism and the mathematical results in the context of a semi-realistic economic capital model of a bank, focusing on the impact of the balance sheet composition on liquidity risk. In addition, we present an algorithm for the computation of the optimal liquidity costs that can be used for applications in practice. In Chapter 4 we introduce some extensions to the basic liquidity risk formalism and discuss their impact on the main results. In Chapter 5 we provide a summary and point out the implications and limitations of the thesis, as well as suggest

possible future research directions.

References

- Acerbi, C. and Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, 8:681–692. 1.3
- Anderson, W., Liese, M., and Weber, S. (2010). Liquidity risk measures. http://www.ims.nus.edu.sg/Programs/financialm09/files/weber_tut1b.pdf. Working paper. 1.3
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent risk measures. *Mathematical Finance*, 9:203–228. 1.1, 4, 1.2, 1.3
- BIS (2000). Sound practices for managing liquidity in banking organisations. Bank of International Settlements. 1.2
- BIS (2006). The management of liquidity risk in financial groups. Bank of International Settlements. 1.2
- BIS (2008a). Liquidity risk: Management and supervisory challenges. Bank of International Settlements. 1.2
- BIS (2008b). Principles for sound liquidity risk management and supervision. Bank of International Settlements. 1.2
- BIS (2009). Range of practices and issues in economic capital frameworks. Bank of International Settlements. 8
- BIS (2010). Basel 3: International framework for liquidity risk measurement, standards and monitoring. Bank of International Settlements. 1.1, 1.2
- Brunnermeier, M., Crocket, A., Goodhart, C., Persaud, A., and Shin, H. S. (2009). The fundamental principles of financial regulation. *Geneva Reports on the World Economy*, 11. 1.1
- Buehlmann, H. (1997). The actuary: the role and limitations of the profession since the mid-19th century. *ASTIN Bulletin*, 27:165–171. 1.1
- Citigroup Inc. (2008). Annual report. <http://www.citibank.com/>. 1.2
- Deutsche Bank (2008). Annual report: Risk report. <http://www.db.com/>. 1.2
- Jarrow, R. A. and Protter, P. (2005). Liquidity risk and risk measure computation. *Review of Futures Markets*, 11:27–39. 1.3
- JPMorgan Chase & Co. (2007). Form 10-k. <http://www.jpmorgan.com/>. 1.2
- Klaassen, P. and van Eeghen, I. (2009). *Economic Capital - How it works and what every manager should know*. Elsevier. 1.1
- Ku, H. (2006). Liquidity risk with coherent risk measures. *Applied Mathematical Finance*, 13:131–141. 1.3
- Lannoo, K. and Casey, J.-P. (2005). Capital adequacy vs. liquidity requirements in banking supervision in the eu. Technical report, Centre for European Policy Studies. 9
- Martin, A. (2007). Liquidity stress testing - scenario modelling in a globally operating bank. In *APRA Liquidity Risk Management Conference*. 1.2
- Matz, L. and Neu, P. (2007). *Liquidity Risk Measurement and Management - A practitioner's guide to global best practices*. Wiley. 1.2
- Morris, S. and Shin, H. S. (2009). Illiquidity component of credit risks. <http://www.princeton.edu/~hsshin/www/IlliquidityComponent.pdf>. Working paper, Princeton University, September. 1.1
- Nikolaou, K. (2009). Liquidity (risk) concepts - definitions and interactions. <http://www.ecb.int/pub/pdf/scpwps/ecbwp1008.pdf>. Bank of England Working Paper, No 1008, February. 1.2
- Standard&Poor's (2004). Fi criteria: Rating banks. <http://www.standardandpoors.com/>. 1.2
- The Goldman Sachs Group (2006). Form 10-k. <http://www2.goldmansachs.com>. 1.2
- UBS AG (2008). Annual report: Risk and treasury management. <http://www.ubs.com/>. 1.2

2

Adjusting EC and RAROC for liquidity risk

A bank's liquidity risk lays in the intersection of funding risk and market liquidity risk. We offer a mathematical framework to make economic capital and RAROC sensitive to liquidity risk. We introduce the concept of a liquidity cost profile as a quantification of a bank's illiquidity at balance sheet level. This leads to the concept of liquidity-adjusted risk measures defined on the vector space of asset and liability pairs. We show that convexity and positive super-homogeneity of risk measures is preserved under the liquidity adjustment, while coherence is not, reflecting the common idea that size does matter. We indicate how liquidity cost profiles can be used to determine whether combining positions is beneficial or harmful. Finally, we address the liquidity-adjustment of the well known Euler allocation principle. Our framework may be a useful addition to the toolbox of bank managers and regulators to manage liquidity risk.

2.1 Introduction

In this chapter, we offer a mathematical framework that makes economic capital and RAROC sensitive to liquidity risk. More specifically, in this chapter we address three issues:

1. Define a sound formalism to make economic capital and RAROC sensitive to liquidity risk, capturing the interplay between a bank's market liquidity risk and funding liquidity risk.
2. Derive basic properties of liquidity-adjusted risk measures with regard to portfolio manipulations and lay the bridge to the discussion in the theory of coherent risk

measures whether subadditivity and positive homogeneity axioms are in conflict with liquidity risk.

3. Clarify the influence of liquidity risk on the capital allocation problem.

Considerable effort has recently been spent on developing formal models that show how to optimally trade asset portfolios in illiquid markets. For an entry to this literature, see for instance Almgren and Chriss (2001); Almgren (2003); Subramanian and Jarrow (2001); Krokmal and Uryasev (2007); Engle and Ferstenberg (2007), and Schied and Schoenborn (2009). While this strand of research is related to our work, these papers focus on sophisticated (dynamic) trading strategies that distribute orders over time to find the optimal balance between permanent, temporary price impacts, and price volatility rather than group level liquidity risk measurement and funding risk.

The recent papers by Jarrow and Protter (2005), Ku (2006), Acerbi and Scandolo (2008), and Anderson et al. (2010) are among the few papers that look at the intersection between liquidity risk, capital adequacy, and risk measure theory, and hence share similar objectives as our paper. Jarrow and Protter (2005) consider the case in which investors are forced to sell a fraction of their holdings portfolio at some risk management horizon instantly and all at once (block sale), incurring liquidity costs relative to the frictionless fair value / MtM value of the portfolio due to deterministic market frictions. In their setting standard risk measures can be adjusted in a straightforward way, leaving the well known coherency axioms (Artzner et al., 1999) in tact. Ku (2006) considers an investor that should be able to unwind its current position without too much loss of its MtM value, if it were required to do so (exogenously determined). The author defines a portfolio as acceptable provided there exists a trading strategy that produces, under some limitations on market liquidity, a cash-only position with possibly having positive future cash flows at some fixed (or random) date in the future that satisfies a convex risk measure constraint. Acerbi and Scandolo (2008) study a framework with illiquid secondary asset markets and “liquidity policies” that impose different forms of liquidity constraints on the portfolio, such as being able to generate a certain amount of cash. The authors stress the difference between values and portfolios. They define “coherent portfolio risk measures” on the vector space of portfolios and find that they are convex in portfolios despite liquidity risk. Anderson et al. (2010) extend the ideas of Acerbi and Scandolo (2008) by generalizing the notion of “liquidity policies” to portfolio and liquidity constraints. However, the authors offer a different definition of liquidity-adjusted risk measures. They define the latter as the minimum amount of cash that needs to be added to the initial portfolio to make it acceptable, which differs from defining risk measures on the liquidity-adjusted portfolio value as Acerbi and Scandolo (2008) do it. As a result they arrive at liquidity-adjusted convex risk measures that are, by construction, “cash invariant”.

Common to all four papers is the idea that a part of an asset portfolio must be liquidated in illiquid secondary asset markets and as a result liquidity costs relative to the frictionless MtM are incurred. Risk measures are consequently defined on the portfolio value less the liquidity costs, except for Anderson et al. (2010). We follow the line of

reasoning of the former papers and also emphasize that liquidity risk naturally changes the portfolio value from a linear to a nonlinear function of the portfolio positions.¹ Despite the similarities with Acerbi and Scandolo (2008) and Anderson et al. (2010), there are important differences between our works. In Acerbi and Scandolo (2008) funding risk is for the most part exogenous. In contrast, we use the concept of asset and liability pairs and we internalize funding risk to some degree with the notion of liquidity call functions. The latter maps asset and liability pairs to random liquidity calls that must be met by the bank on short notice by liquidating part of its asset portfolio without being able to rely on unsecured borrowing (interbank market). This is similar to Anderson et al. (2010)'s short-term cash flow function. By imposing a liquidity call constraint we can investigate the optimization problem as well as emphasize Type 2 liquidity risk. Of the above papers, we are the only one who stress the effects of Type 2 liquidity risk on concepts such as risk diversification and capital requirements, which turns out to be of importance. In addition, we also discuss the problem of the allocation of liquidity-adjusted economic capital and RAROC to business units, which none of the above papers do.

The chapter is organized as follows: in Section 2.2 we introduce the basic concept of optimal liquidity costs. In Section 2.3 we characterize the optimization problem and define the concept of liquidity cost profiles. In Section 2.4 we define liquidity-adjusted EC and RAROC on the space of portfolios and discuss some interpretation issues of using liquidity-adjusted EC to determine a bank's capital requirements. In Section 2.5 we introduce asset and liability pairs and liquidity call functions. We use these to derive some basic properties of liquidity-adjusted risk measures defined now on the space of asset and liability pairs under liquidity call functions. In Section 2.6 we address the capital allocation problem under liquidity risk. In Section 2.7 we sketch some of the problems related to calibrating liquidity risk models. In Section 2.8 we illustrate the main concepts of the chapter in a simulation example. We conclude with a discussion in Section 2.10.

2.2 Mathematical framework

Consider a market in which $N + 1$ assets or asset classes or business units are available, indexed by $i = 0, 1, \dots, N$, where $i = 0$ is reserved for cash (or near-cash).² Suppose banks are endowed with an *asset portfolio*. At this stage, there is no need to specify whether this reflects the current position of a bank, or a hypothetical future position under a certain scenario that is considered.

Definition 2.1 Asset portfolio. An *asset portfolio* p is a $N + 1$ nonnegative real-valued vector $p = (p_0, \dots, p_N) \in \mathcal{P} = \mathbb{R}_+^{N+1}$, where p_0 denotes the *cash position*.

¹Typically the portfolio value is a nonlinear function of the risk factors but a linear function of the portfolio positions (see, e.g., Chapter 2 in McNeil et al. (2005)).

²We will use the term asset and business unit interchangeably throughout the text. Our formalism applies to both interpretations.

The p_i may be interpreted as the number of a particular asset contracts or the amount of currency invested in that asset (business unit). Now suppose the bank needs to generate a certain amount α in cash, e.g., from a sudden fund withdrawals, *only* from liquidating its assets p . We call this cash obligation a *liquidity call*. Short-selling assets³ or generating cash from extra unsecured borrowing is not allowed.⁴ Of course, if it were allowed, banks would not face a liquidity crisis as they could easily meet α . Note that one could always interpret α as being the liquidity call that is left after unsecured funding channels have been exhausted. The most straightforward way to withstand a liquidity call at level $\alpha \in \mathbb{R}_+$ is to have the amount available in cash, i.e., to have a portfolio $p \in \mathcal{P}$ such that $p_0 \geq \alpha$. However, while having large amounts of cash at all times is safe, the opportunity costs usually are prohibitive. As a result, it is reasonable to assume that the bank needs to liquidate parts of its asset portfolios to meet α .

We first consider, as a reference point, the proceeds of selling assets in a frictionless market. We refer to these proceeds as the fair value or Marked-to-Market/Marked-to-Model (MtM) value of a portfolio.⁵

Definition 2.2 MtM valuation function. Let $V_i \geq 0$ for $i = 0, \dots, N$ be the fair asset prices. The *MtM valuation function* is a linear function $V : \mathcal{P} \rightarrow \mathbb{R}_+$ given by $V(p) := p_0 + \sum_{i=1}^N p_i V_i$.

However, we commonly observe market frictions in secondary asset markets, especially in times of turbulences.⁶ We formalize market frictions by way of proceed functions.

Definition 2.3 Asset proceed function. An *asset proceed function* for asset i is a non-decreasing, continuous concave function $G_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that satisfies for all $x_i \in \mathbb{R}_+$ and for all $i > 0$, $G_i(x_i) \leq x_i V_i$ and $G_0(x_0) = x_0$. The space of all asset proceed functions is denoted by \mathcal{G} .

Monotonicity and concavity is justified reasonably well by economic intuition and are not very restrictive. Furthermore, they are in line with theoretical market microstructure literature (Glosten and Milgrom, 1985; Kyle, 1985), recent limit order book modeling (Alfonsi et al., 2010), and empirical analysis (Bouchaud, 2009). As fixed transaction costs are negligible in our context, continuity of the asset proceed functions follows

³We do not consider short positions because we do not think it is relevant for our purpose of liquidity risk management on group level and would only lead to unnecessary complications. However, extending our framework in this direction is possible.

⁴While the assumption of no access to unsecured funding is quite pessimistic, banks have commonly assumed even before the recent crisis that unsecured borrowing is not available during crisis time in their liquidity stress analyses (Matz and Neu, 2007). In addition, the stress scenario used under Basel III for the LCR assumes that a bank's funding ability is severely impaired. However, more importantly we have witnessed it happen during the recent Subprime crisis.

⁵"Marking-to-market" and "fair values" are often used as synonyms. However, fair value is a more general concept than MtM as it does not depend on the existence of active markets with determinable market prices as MtM does. Having said that, we will use the term MtM and fair value interchangeably.

⁶In this paper we take the existence of secondary asset market frictions as *given* and do not attempt to explain it from more basic concepts.

from the assumption of continuity in zero. Note that finite market depth can formally be represented by constant proceeds beyond some particular transaction size. We assume that cash is a frictionless asset and has a unit price. We do not formalize the notion of buying assets. We could, however, extend our framework in this direction (cf., Jarrow and Protter, 2005; Acerbi and Scandolo, 2008).⁷

We assume that the proceeds of liquidating more than one asset at a time is simply the sum of the individual proceeds.

Definition 2.4 Portfolio proceed function. The *portfolio proceed function* is a function $G : \mathcal{P} \rightarrow \mathbb{R}_+$ given by $G(x) := \sum_{i=0}^N G_i(x_i) = x_0 + \sum_{i=1}^N G_i(x_i)$.

By taking the sum, we do not allow that liquidating one asset class has an effect on the proceeds of liquidating another asset class. We do not formalize such *cross-effects* here because we believe they would only distract from the main idea without adding conceptual insights. However, we discuss the consequences of allowing them in Chapter 4. For a treatment of cross-effects we refer the interested reader to Schoenborn (2008).

Comparing the proceeds to the MtM value leads to a natural definition of the liquidity costs associated with the liquidation of a portfolio.

Definition 2.5 Liquidity cost function. The *liquidity cost function* is a function $C : \mathcal{P} \rightarrow \mathbb{R}_+$ given by $C(x) := V(x) - G(x)$.

We collect some basic properties of the portfolio proceed function and the liquidity cost function in the following lemma.

Lemma 2.6. *Let G be a portfolio proceed function and C a liquidity cost function. Then*

1. *both G and C are non-decreasing, continuous, zero in zero, and $G(x), C(x) \in [0, V(x)]$ for all $x \in \mathcal{P}$.*
2. *G is concave, subadditive, and sub-homogenous: $G(\lambda x) \leq \lambda G(x)$ for all $\lambda \geq 1$ and all $x \in \mathcal{P}$.*
3. *C is convex, superadditive, and super-homogenous: $C(\lambda x) \geq \lambda C(x)$ for all $\lambda \geq 1$ and all $x \in \mathcal{P}$.*

Proof of Lemma 2.6. It follows directly that $G(0) = 0$ that G is concave (the nonnegative sum of concave functions is concave) and that G is non-decreasing. Sub-homogeneity follows from concavity. For subadditivity consider the case for the asset proceed function $G_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ first. Using sub-homogeneity, we have for $a, b \in \mathbb{R}_+$ that $G_i(a) + G_i(b) = G_i((a+b)\frac{a}{a+b}) + G_i((a+b)\frac{b}{a+b}) \geq \frac{a}{a+b}G_i(a+b) + \frac{b}{a+b}G_i(a+b) = G_i(a+b)$. The result follows because G is simply the sum of individual asset proceed functions. From Definition 2.5 it follows that $C(0) = 0$ and that C is convex and nonnegative, hence non-decreasing. The other claims follow directly. \square

⁷The asset proceed function may also be interpreted as the process of repoing asset with a transaction size dependent haircut. However, for it to make sense in our setting, we need to be willing to accept that the encountered value loss is a realized loss. Under current accounting standards such a loss is generally not recognized but we note that these issues are currently under revision by the IASB.

Denote by \mathcal{L}^α the set of all portfolios from which it is possible to generate at least α cash by liquidating assets without short-selling assets or using unsecured borrowing facilities.⁸

Definition 2.7 The liquidity feasibility set. Given a liquidity call $\alpha \in \mathbb{R}_+$ and proceed functions $G_i \in \mathcal{G}$ for $i = 0, 1, \dots, N$, the *liquidity feasibility set* is defined by $\mathcal{L}^\alpha := \{p \in \mathcal{P} \mid G(p) \geq \alpha\}$ with G as defined in Definition 2.4.

For expository purposes we postpone imposing more structure on α to Section 2.5. For now it is sufficient to take it as an exogenously given object.

In the following definition, which is very similar to ideas in Acerbi and Scandolo (2008), we introduce the *optimal liquidity cost function* which assigns costs for a given liquidity call to a portfolio. It handles Type 1 ($p \in \mathcal{L}^\alpha$) and Type 2 ($p \notin \mathcal{L}^\alpha$) liquidity risk and it is the key concept in our framework.

Definition 2.8 Optimal liquidity cost function. The *optimal liquidity cost function* for a given liquidity call $\alpha \in \mathbb{R}_+$ is the function $C^\alpha : \mathcal{P} \rightarrow \mathbb{R}_+$ given by⁹

$$C^\alpha(p) := \begin{cases} \inf\{C(x) \mid 0 \leq x \leq p \text{ and } G(x) \geq \alpha\}, & \text{for } p \in \mathcal{L}^\alpha \\ V(p), & \text{for } p \notin \mathcal{L}^\alpha. \end{cases}$$

It is immediately clear that we are allowed to write min instead of inf because the domain of the infimum is nonempty and compact and C is continuous. Furthermore, if the optimal liquidity costs $C^\alpha(p)$ are nonzero, the equality $G(x^*) = \alpha$ must hold for the optimal liquidation strategy x^* , because otherwise down-scaling (always possible due to continuity of G) would yield less costs. In the trivial case where costs are zero, we can still impose $G(x) = \alpha$ without loss of generality. Hence, we can from now on use

$$C^\alpha(p) = \begin{cases} \min\{C(x) \mid 0 \leq x \leq p \text{ and } G(x) = \alpha\}, & \text{for } p \in \mathcal{L}^\alpha \\ V(p), & \text{for } p \notin \mathcal{L}^\alpha. \end{cases}$$

Note that we are dealing with a convex optimization problem. Hence, any local optimum is a global optimum and the set of all optimal liquidation strategies is a convex subset of $\{x \in \mathcal{P} \mid 0 \leq x \leq p\}$.¹⁰

The intuition behind the definition is reasonably straightforward: the optimization problem approximates the real life problem a bank would need to solve in case of an

⁸Readers familiar with Acerbi and Scandolo (2008) should not confuse our \mathcal{L}^α with their “cash liquidity policies”, given by $\mathcal{L}(\alpha) := \{p \in \mathcal{P} \mid p_0 \geq \alpha\}$.

⁹We write $x \leq y$ for $x, y \in \mathbb{R}^n$ if $x_i \leq y_i$ for $i = 1, \dots, n$.

¹⁰A standard result of minimizing a convex function over a convex set. Suppose we have for a given $\alpha \in \mathbb{R}_+$ and $p \in \mathcal{P}$ two optimal liquidation strategies x_1 and x_2 , $x_1 \neq x_2$. As a result, we have that $C^\alpha(p) = C(x_1) = C(x_2)$. Now consider $x_\tau := \tau x_1 + (1 - \tau)x_2$ for some $\tau \in (0, 1)$. By convexity of the feasibility set we know that x_τ is feasible as well and using the convexity of C , the fact that $C(x_1)$ must be globally optimal, and the fact that $C(x_1) = C(x_2)$ we have that $C(x_1) \leq C(x_\tau) \leq \tau C(x_1) + (1 - \tau)C(x_2) = C(x_1)$, and hence $C(x_1) = C(x_\tau)$.

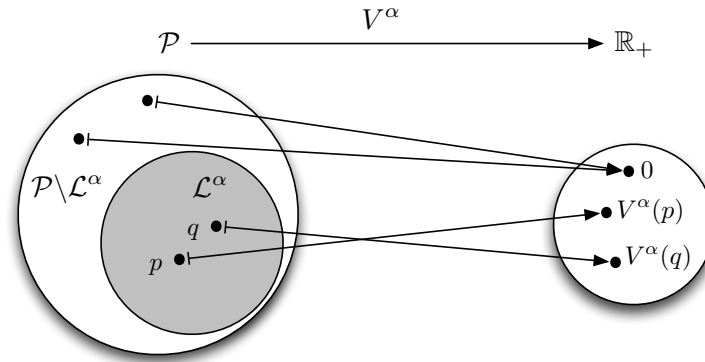


Figure 2.1: Visual illustration of the liquidity-adjusted valuation function V^α defined in Definition 2.9. In particular, the figure demonstrates the workings of the 100% with help of the portfolios outside of \mathcal{L}^α .

idiosyncratic liquidity crisis. Note that we allow, for simplicity, *infinite divisibility* of positions in the optimization problem. For the case the bank portfolio is illiquid (Type 2 liquidity risk), i.e., $p \notin \mathcal{L}^\alpha$, we say that all asset value is lost because default is an absorbing state. This treatment of illiquid states deviates from Acerbi and Scandolo (2008) and Anderson et al. (2010) as they set the “costs” to ∞ in case $p \notin \mathcal{L}^\alpha$. Their approach is the common way to treat *hard constraints* in optimization problems. We choose differently because we believe there are some advantages in mapping the default by illiquidity to a value loss as will be explained in later sections. Note that the optimal liquidity costs under a zero liquidity call is zero for all portfolios: $(\forall p \in \mathcal{P}) C^0(p) = 0$.

Closely related to Definition 2.8 is the concept of the *liquidity-adjusted value* of a portfolio:

Definition 2.9 Liquidity-adjusted valuation function. The *liquidity-adjusted valuation function* for a given $\alpha \in \mathbb{R}_+$ is a map $V^\alpha : \mathcal{P} \rightarrow \mathbb{R}_+$ such that the liquidity-adjusted value of a $p \in \mathcal{P}$, given a liquidity call $\alpha \in \mathbb{R}_+$, is given by $V^\alpha(p) := V(p) - C^\alpha(p)$.

In Figure 2.1 we illustrate the map visually. Notice that we do not consider *permanent price impacts* as we value the remaining portfolio at the frictionless MtM value. The idea that the own liquidation behavior can leave a permanent mark on the asset price is known as permanent price impact and becomes important in situations where one distributes large orders over time (Almgren and Chriss, 2001; Almgren, 2003) or considers contagion of price shocks via MtM accounting (Plantin et al., 2008). We do not formalize these effects here because we believe they would distract from the main idea without adding conceptual insights. However, we refer the interested reader to Chapter 4 for a discussion of their impact on our framework.

Remark 2.2.1. The implied constrained optimal liquidation problem is static and does not consider timing issues. In reality, generating cash from assets is not instantaneously as it takes time depending on the market and the asset (class). However, integrating different liquidation periods for different asset (classes) into the standard static setting is problematic (see, e.g., p. 41 in McNeil et al., 2005). We refer readers to Brigo and

Nordio (2010) for a constructive approach. Also, liquidity calls do not arise instantaneously but are rather spread over time. While we are aware of these issues, we do not explicitly formalize them. We can only indirectly include them in our framework by interpreting, e.g., α as a cumulative liquidity call over some time interval. There is a clear resemblance between our α and the total net cash outflow over 30 days under stress in the context of the LCR in Basel III. On this issue, we would like to echo Jarrow and Protter (2005)'s argument in favor of keeping it simple to support model transparency and robustness on this issue.

Remark 2.2.2. We do not claim that the liquidity-adjusted portfolio value is a suitable alternative to MtM valuation in the accounting context. The liquidity costs will “live” entirely in the future as will be made clear in subsequent sections. The reason for this is that we would have problems with interpreting and specifying cash requirements at time zero. Also, even if we could get around that, it is unclear, whether it would eliminate any of the potential disadvantages associated with MtM valuation as discussed, e.g., in Allen and Carletti (2008).

Remark 2.2.3. It is possible to include without much difficulties also other side constraints into the optimization problem in Definition 2.8. Other constraints could extend the framework to handle non-cash obligations but we believe that cash obligations remain the most obvious choice in the context of liquidity risk of banks. See, e.g., Anderson et al. (2010) for an extension in this direction.

2.3 Liquidity cost profile

In this section we characterize the optimization problem and show how this gives rise to the concept of a liquidity cost profile of a portfolio. We will use these results in Section 2.5 for the characterization of the optimal liquidity cost function and liquidity-adjusted risk measures.

As preparation for the main result, we introduce some notation related to the partial derivatives of the portfolio proceed function G . From the properties of the asset proceed functions G_i (Definition 2.3) it follows that their left and right derivatives

$$G'_i(x_i^-) := \lim_{h \nearrow 0} \frac{G_i(x_i + h) - G_i(x_i)}{h}$$

$$G'_i(x_i^+) := \lim_{h \searrow 0} \frac{G_i(x_i + h) - G_i(x_i)}{h}$$

exist and are both non-decreasing functions, taking values in $[0, V_i]$ and differ in at most a countable set of points, exactly, those points where both have a downward jump. Hence, G_i is continuously differentiable almost everywhere and so is the portfolio proceed function G . Note that by Definition 2.4 we know that the partial derivative G with respect to the i th position equals the derivative of G_i . The following characterization of optimality is easily obtained from standard variational analysis.

Lemma 2.10. *Given an asset portfolio $p \in \mathcal{P}$, a liquidity call $\alpha \in \mathbb{R}_+$, proceed functions*

$G_i \in \mathcal{G}$ for $i = 1, \dots, N$, and a liquidation strategy $x \in \mathcal{P}$ generating α cash (cf. Definition 2.8). Then x is optimal, if and only if there exists a $\mu_p(\alpha) \in [0, 1]$ such that for $i = 1, \dots, N$,

$$G'_i(x_i^-) \geq \mu_p V_i \text{ or } x_i = 0 \quad (2.1)$$

$$G'_i(x_i^+) \leq \mu_p V_i \text{ or } x_i = p_i. \quad (2.2)$$

In particular,

$$G'_i(x_i) = \mu_p V_i \quad (2.3)$$

for all i with $x_i \in (0, p_i)$ and G_i differentiable in x_i . For almost all $\alpha \in (0, V(p))$, $\mu_p(\alpha)$ is unique and Equation 2.3 applies to at least one i where $x_i \in (0, p_i)$.

Proof of Lemma 2.10. Necessity of (2.1) and (2.2): Let i, j denote a pair of indices for which $x_i > 0$ and $x_j < p_j$ (if no such pair exists, either $x_k = 0$ or $x_k = p_k$ for all but at most one index k , and necessity of (2.1) and (2.2) is easily verified in these simple cases). Now consider a variation of x that amounts to exchanging a small amount $\delta > 0$ of assets i for ϵ (extra) assets j (changing θ into $\theta + \delta e_i + \delta - \epsilon_j e_j$). Such a variation is admissible if $x_i > 0$, $x_j < 1$ and if α cash is still generated, so $\delta G'_i(x_i^-) \approx \epsilon G'_j(x_j^+)$, which means that

$$\epsilon = \frac{G'_i(x_i^-)}{G'_j(x_j^+)} + h.o.t.$$

Then x can only be optimal if a (small) variation in this direction is not decreasing the liquidation costs, i.e., it must hold that $\epsilon(V_j - G'_j(x_j^+)) \geq \delta(V_i - G'_i(x_i^-)) > 0$. Substituting the expression for ϵ yields that $G'_i(x_i^-)/V_i \geq G'_j(x_j^+)/V_j$ for any such pair i, j . Now define $\mu^- := \min\{G'_i(x_i^-)/V_i \mid i \text{ such that } x_i > 0\}$ and $\mu^+ := \max\{G'_j(x_j^+)/V_j \mid j \text{ such that } x_j < p_j\}$. It follows that $\mu^- \geq \mu^+$, and we can choose (any) μ_p within or at these bounds. It is clear that (2.3) follows from (2.1) and (2.2), so this is also a necessary condition for optimality of x .

To prove sufficiency of (2.1) and (2.2), let μ_p satisfy these conditions for a given x . Consider another strategy y with $G(y) = \alpha$. For all i with $y_i \leq x_i$, the extra proceeds are bounded by $(y_i - x_i)\mu_p V_i$, while for all i with $y_i > x_i$, the reduction in proceeds is at least $(x_i - y_i)\mu_p V_i$. From $G(y) = \alpha = G(x)$ it follows that the extra proceeds cancel against the reductions, implying that $\sum_i (y_i - x_i)\mu_p V_i \geq 0$, and hence that y is at least as costly as x . So x is optimal.

Note that (2.3) can also be derived as follows: recall the constraint optimization problem

$$\min\{V(x) - G(x) \mid G(x) = \alpha \text{ and } x \in \Pi_p\},$$

where the range Π_p denotes the set of all liquidation policies that do not involve short-selling, which can be parameterized by a vector θ containing the fraction of each asset used in liquidation, i.e., $\Pi_p = \{\theta p \mid 0 \leq \theta \leq 1\}$. The corresponding Lagrangian function is given by

$$\Lambda(x, \lambda) = V(x) - G(x) + \lambda(\alpha - G(x)),$$

where λ denotes the Lagrange multiplier. The first order conditions (FOCs) are

$$\frac{dV(x)}{dx_i} - \frac{dG(x)}{dx_i} = \lambda \frac{dG(x)}{dx_i} \text{ for all } i,$$

hence for all i ,

$$\lambda = \frac{\frac{dV(x)}{dx_i}}{\frac{dG(x)}{dx_i}} - 1.$$

Let V_i denote the fair unit price of asset i , then $\frac{dV(x)}{dx_i} = V_i$, and we can rewrite the FOCs, using the notation as introduced above, as follows

$$\mu_i := \frac{G'_i(x_i)}{V_i} = \frac{1}{1 + \lambda} =: \mu$$

As discussed earlier G is continuously differentiable, hence it follows immediately that the conditions are necessary for optimality, and in fact the third equation holds. Sufficiency follows from the fact that G' is non-increasing by assumption. \square

We can interpret the figure $\mu_p(\alpha)$ as the marginal liquidity proceeds of a portfolio under a given liquidity call, expressed as amount of cash per liquidated MtM-value. It is an upper bound for the marginal cash return in liquidating the next bit of cash, $\mu_p(\alpha^+)$. The lemma states that optimal liquidation amounts to using all assets up to one and the same level of marginal liquidity proceeds μ_p , if possible; otherwise the asset is either used completely, if it never reaches that level, or it is not used at all, if its liquidity costs for arbitrary small volumes is already too high. It turns out to be more convenient to work with marginal liquidity costs per generated unit of cash, i.e., $\lambda_p(\alpha) := \frac{1 - \mu_p(\alpha)}{\mu_p(\alpha)}$. We refer to this function as the *liquidity cost profile* of a portfolio.

Definition 2.11 Liquidity cost profile. The *liquidity cost profile* of portfolio p is the unique left-continuous function $\lambda_p : (0, G(p)] \rightarrow \mathbb{R}_+$ so that for all α , $\mu_p(\alpha) := \frac{1}{1 + \lambda_p(\alpha)}$ satisfies the conditions in Equation 2.3.

It is easily verified that the optimal liquidation costs of a portfolio p for generating α cash is given by

$$C^\alpha(p) = \begin{cases} \int_0^\alpha \lambda_p(s) ds, & \text{for } p \in \mathcal{L}^\alpha \\ V(p), & \text{for } p \notin \mathcal{L}^\alpha. \end{cases}$$

Example 2.12 Liquidity cost profile. Assume that $N = 2$ and consider the asset portfolio $p = (p_0, p_1, p_2) = (10, 10, 10)$. Proceed functions take an exponential functional form: $G_i(x_i) = (V_i/\theta_i)(1 - e^{-\theta_i x_i})$, $i = 1, 2$, where θ_i is a friction parameter and we have that $\theta_1 > \theta_2$. Given that $G'_i(x_i) = V_i e^{-\theta_i x_i}$, we can show using Lemma 2.10 that the liquidity cost profile for this example is given by

$$\lambda_p(\alpha) = \begin{cases} 0 & \text{for } 0 \leq \alpha \leq p_0, \\ \hat{\lambda}_p(\alpha) = -\frac{p_0 - \alpha}{V_1/\theta_1 + V_2/\theta_2 + p_0 - \alpha} & \text{for } p_0 < \alpha \leq \bar{\alpha}, \\ \bar{\lambda}_p(\alpha) = -\frac{p_0 + G_2(p_2) - \alpha}{V_1/\theta_1 + p_0 + G_2(p_2) - \alpha} & \text{for } \bar{\alpha} < \alpha \leq G(p), \end{cases}$$

where $\bar{\alpha} = G(p_0, \theta_2 p_2 / \theta_1, 10) = 117.15$, which marks the point at which Asset 2 is used up completely. The optimal liquidity costs are then given by

$$C^\alpha(p) = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq p_0 \\ \int_{p_0}^{\alpha} \hat{\lambda}_p(s) ds & \text{if } p_0 < \alpha \leq \bar{\alpha} \\ \int_{p_0}^{\bar{\alpha}} \hat{\lambda}_p(s) ds + \int_{\bar{\alpha}}^{\alpha} \bar{\lambda}_p(s) ds & \text{if } \bar{\alpha} < \alpha \leq G(p) \\ V(p) & \text{if } \alpha > G(p). \end{cases}$$

Suppose we have the following parameter values: $p = (10, 10, 10)$, $V_1 = 10$, $V_2 = 8$, $\theta_1 = 0.08$, $\theta_2 = 0.04$ and $\alpha = 130$. The optimal liquidity costs for these parameters are $C^{130}(p) = 30.82$ and the liquidity adjusted value is $V^{130}(p) = 190 - 30.82 = 159.18$. In Figure 2.2 we plot the main functions for the example portfolio and different friction parameter pairings, and illustrate the integral form of the optimal liquidity costs for the above example parameters. ■

Remark 2.3.1. It is easily verified, using Lemma 2.10, that the “common-sense” strategy of liquidating first the most liquid asset, then after its position is exhausted start liquidating the second most liquid asset, and so on for the whole asset portfolio is generally not optimal (see Duffie and Ziegler (2003) for an example of its use in a different context). It is, however, the optimal liquidation strategy for the special case of linear proceed functions as the partial derivatives are constant. See Example 3.4 on p. 108 in Chapter 2 for a more detailed discussion.

2.4 Economic capital and RAROC with liquidity risk

In this section we bring the concepts of optimal liquidity costs and liquidity cost profiles into the standard static risk measurement setting and define liquidity-adjusted EC and RAROC, as well as discuss some issues surrounding the interpretation of liquidity-adjusted EC as a capital requirement.

2.4.1 Capital adequacy assessment

Bank managers, supervisors, and debt holders are interested in ensuring the continuity of a bank and hence avoiding the bank’s insolvency is of major interest to them. As insolvency occurs when the value of the assets drop below the value of the liabilities, the bank’s actual capital acts as a buffer against future, unexpected value losses, assuming that expected losses are covered by margins and provisions. Consequently, comparing the risks of the bank’s asset portfolio to the bank’s actual amount of loss-absorbing available capital is crucial. We refer to this process as assessing the *capital adequacy* of a bank.

The value of available loss-absorbing capital of the bank depends on who is assessing the adequacy. For instance, debt holders should only take into account the actual amount of the bank’s loss-absorbing available capital that is junior to their claims.

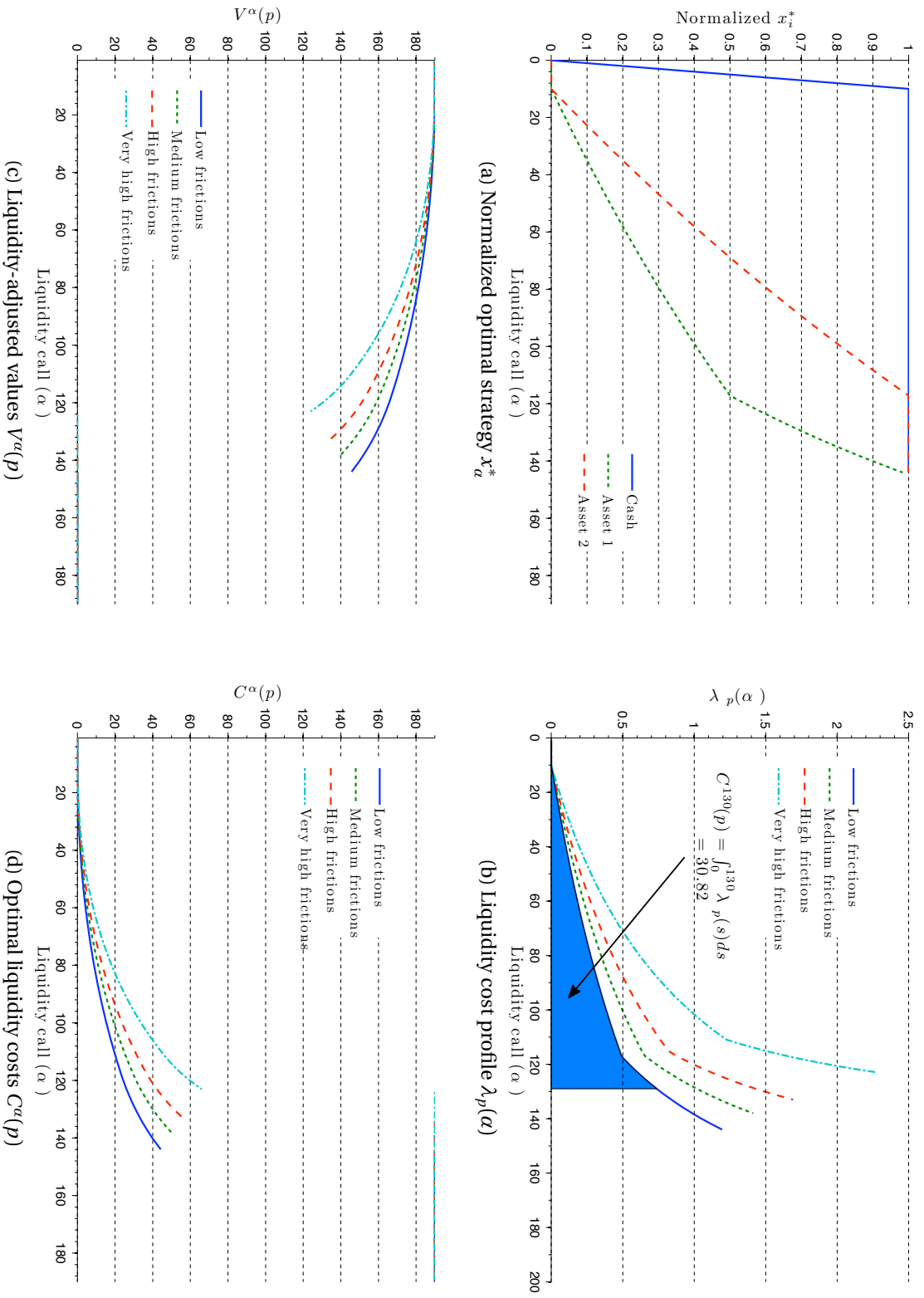


Figure 2.2: The normalized optimal liquidation strategy ($x_i^* := x_i(\alpha)/p_i$) as a function of α for the market liquidity parameter pair ($\theta_{1,1} = 0.08, \theta_{1,2} = 0.04$) (upper left), the liquidity cost profile (upper right), the liquidity-adjusted value (lower right), and the optimal liquidity costs (lower right) as a function of the liquidity call α for four different market liquidity parameter pairs: ($\theta_{1,1} = 0.08, \theta_{1,2} = 0.04$), ($\theta_{2,1} = 0.09, \theta_{2,2} = 0.05$), ($\theta_{3,1} = 0.1, \theta_{3,2} = 0.06$), and ($\theta_{4,1} = 0.12, \theta_{4,2} = 0.08$), labeled “Low frictions”, “Medium frictions”, “High frictions”, and “Very High frictions”, respectively.

Assessing the risk of unexpected value losses of the bank is a difficult task. Usually one approaches the problem by asking the question what the amount of capital the bank *ought* to have *today* in order to limit the possibility of default (insolvency) within a given time-frame, assuming the bank's portfolio composition is fixed. Using the terminology of Jarrow and Purnanandam (2005) we refer to it as the *capital determination* problem, and note that it is related but different from the *capital budgeting* problem, which deals with the bank's problem of choosing the composition of its assets and liabilities so as to maximize its risk/return trade-off, subject to any regulatory capital requirements.

In practice, the capital determination and capital adequacy assessment procedure is straightforward. Given a probability distribution for the bank's profit and loss (P&L) at the risk management horizon T , one computes the quantile (VaR) at a confidence level¹¹ and takes the result, say $b \in (0, \infty)$, as the amount of capital the bank *ought* to have today to be considered adequately covered against large unexpected value losses at time T . Furthermore, if b is smaller or equal than the actual loss-absorbing capital of the bank today, then the bank passes the capital adequacy test and no *corrective actions* are required. If, however, b is larger than the actual capital level of the bank today, the bank fails the test and has to engage in corrective actions.

Commonly, the regulator's estimate of what capital the bank ought to have is referred to as *regulatory capital* (RC) and the bank's internal estimate is referred to as *economic capital* (EC). EC is an important *management control tool* used within a bank. It is not only determined at group level but also allocated to business units, sub-portfolios, and products. Furthermore, it has now become a standard in the banking sector to measure performance by a risk-adjusted ratio of return over capital (RAROC), with (allocated) EC taken as denominator and expected P&L in the numerator. As mentioned earlier, we will use the term EC instead of RC to abstract from the functional form of RC after Basel II.

2.4.2 Economic capital and RAROC

Since the publication of Basel I and II banks have put tremendous effort in developing models for RC and EC. The elementary component of an EC model is a set of scenarios derived from a stochastic model and/or historical simulation that is often complemented with extra stress scenarios imposed by regulators or risk managers. These scenarios, together with their probabilities or assigned weights, determine a probability distribution of the annual P&L. Banks use this distribution to determine the EC corresponding to a quantile at a certain confidence level or more generally the outcome of a risk measure.

We consider, for simplicity, two moments in time: today denoted by $t = 0$ and some risk management horizon denoted by $t = T$ (usually taken to be 1 year). We assume that a bank's position today leads to overall P&L at time T , denoted by X . This

¹¹The confidence level is typically related to the bank's target credit rating in such a way that the implied PD is in line with historical default rates in the industry.

models the future P&L as a random variable X at T , where $X(\omega)$ represents the profit ($X(\omega) > 0$) or loss ($X(\omega) < 0$) at time T , if the scenario ω realizes. More formally, fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathcal{H} := L^1(\Omega, \mathcal{F}, \mathbb{P})$ the space of all integrable random variables on (Ω, \mathcal{F}) .¹² We interpret \mathcal{H} as the space of all conceivable future P&L of a bank over the time horizon T . Risk measures are defined as mappings from this space to the real line: $\rho : \mathcal{H} \rightarrow \mathbb{R}$. In other words, a risk measure maps a given risk $X \in \mathcal{H}$ to its level of riskiness $\rho(X)$, expressed as a real number.

In practice, the risk measure of choice for EC is Value at Risk (VaR) at a certain confidence level, i.e., the quantile, but mathematically any *monetary risk measure* is an admissible candidate for determining the EC of a bank.

Definition 2.13 Monetary risk measure. A risk measure $\rho : \mathcal{H} \rightarrow \mathbb{R}$ is called a *monetary risk measure* if $\rho(0) = 0$ and, in addition, satisfies the following properties:

TA Translational anti-variance:

$$\text{for all } X \in \mathcal{H} \text{ and for all } e \in \mathbb{R}, \rho(X + e) = \rho(X) - e.$$

AM Anti-monotonicity:

$$\text{for all } X, Y \in \mathcal{H} \text{ and } X \geq Y, \rho(X) \leq \rho(Y).$$

Translational anti-variance is important because it is tightly linked to the capital determination idea. Under some mild technical assumptions, one can show that the outcome of a translational anti-variant risk measure can be interpreted as the minimum capital a bank needs to have zero risk, as specified by the risk measure. To see this, consider the following result of Artzner et al. (1999) (Proposition 2.2),

$$\rho(X) = \inf\{m \in \mathbb{R} \mid \rho(X + m) \leq 0\} \quad (X \in \mathcal{H}). \quad (2.4)$$

Equation 2.4 tells us that $\rho(X)$ is equal to the minimum amount m that needs to be added to the P&L X at time the time T , i.e., be available in every scenario, so as to have zero risk (as specified by the risk measure). Now the real number m , when positive (the typical case),¹³ can be interpreted as the minimum amount of capital a bank needs at time T as m “absorbs” potential losses per scenario. Furthermore, as we assume that capital levels remain unchanged between today and at time T , $\rho(X)$ determines the minimum amount of capital needed at time zero as well. In that sense it generalizes the practical capital determination procedure, which relies on VaR because any translational anti-variant risk measure allows this interpretation. Of course, there are other, stronger properties than translation anti-variance that we may demand of a risk measures. We turn towards these properties in Section 2.5.

Before we can properly define the EC of a bank in our setting, we need to clarify the relationship between a bank’s initial portfolio $p \in \mathcal{P}$ and the bank’s overall P&L variable X at T . It is standard to assume that there is a *linear* relationship between the exposure

¹²In some cases we can work with $L^0(\Omega, \mathcal{F}, \mathbb{P})$, e.g., when we are dealing with VaR, but for simplicity we stick with \mathcal{H} in this thesis.

¹³Of course, technically it is possible that m is negative, which is a bit more difficult to interpret in practical terms.

p and the individual unit P&L of an asset or business unit X_0, \dots, X_N , so that the overall portfolio risk is the sum of individual risks weighted by the exposures. Formally, we can express this as follows:

$$X(p) := \sum_{i=0}^N p_i X_i \quad X_i \in \mathcal{H} \text{ for } i = 0, \dots, N. \quad (2.5)$$

Note that by definition we have that $X(p) \in \mathcal{H}$. The linearity assumption makes sense in the typical specifications of market risk (trading book) and credit risk (banking book) but is not entirely convincing for operational risk because it is not a “position risk”. While it is reasonable to assume that operational losses increase with position size, i.e., larger banks have a chance of higher losses, saying that losses scale linearly with the business unit size is more difficult to justify. However, under the Basic Indicator Approach and the Standardized Approach of Basel II, capital charges for operational losses scale linearly with a bank’s overall and business line gross income, respectively. For simplicity we will assume from now on that linearity holds in the absence of liquidity risk.

The definition in Equation 3.3 requires some explanation regarding the connection between p and time. In the subsequent analysis, we simply assume that we do as if we confront the bank’s current position p with scenarios modeled as occurring at T . That way the position p is the same today as it is at T . The assumption is simplistic but standard. A more general and conceptually more satisfying approach would be to model the bank’s position as a random variable. Such an approach would allow us to factor in realistic reactions of the bank and bank’s investors to scenarios with regard to replacing maturing assets and liabilities, portfolio growths, reductions of exposures, reclassifications out of the *Fair Value through Profit and Loss* (FVTPL) / *Available-for-sale* (AFS) category as seen during the Subprime crisis etc. Unfortunately, systemically formalizing such behavioral assumptions is difficult and hence we stick to the simplistic perspective.¹⁴

Combining Equation 3.3 with the concept of a monetary risk measure, we arrive at the formal definition of EC.

Definition 2.14 Economic capital. Given a monetary risk measure ρ and $X \in \mathcal{H}$, the *economic capital* (EC) is a map $EC : \mathcal{P} \rightarrow \mathbb{R}$ given by $EC(p) := \rho(X(p))$.¹⁵

Note that we do not explicitly refer to the monetary risk measure ρ in the notation of EC, e.g., by using an index like $EC_\rho(p)$. Instead whenever needed we refer to this risk measure as the *underlying* (monetary) risk measure.

The portfolio RAROC is defined as the ratio of the expected P&L to the EC.

¹⁴At the same time, it should be noted that the current financial regulation allows the use of so called *management intervention* strategies in models in the context of market risk and interest rate risk. However, even these cases the exact specification is difficult and sometimes controversial.

¹⁵In practice one usually considers for credit risk the unexpected loss for EC computations, i.e., $\bar{EC}(p) := \rho(X(p) - E[X(p)])$, which leads to $\rho(X(p)) + E[X(p)]$ for monetary risk measures. We do not consider this case here because it does not change our results and would only increase the notational burden.

Definition 2.15 RAROC. Given a monetary risk measure ρ and $X \in \mathcal{H}$, the *risk adjusted return on capital* (RAROC) is a map $\text{RAROC} : \mathcal{P} \rightarrow \mathbb{R}$ given by $\text{RAROC}(p) := \frac{E[X(p)]}{\text{EC}(p)}$.

2.4.3 Liquidity adjustment

For incorporating liquidity costs into the EC and RAROC machinery, we make the simplifying assumption that the P&L of the bank without liquidity risk is given by the fair value change of the asset portfolio:

$$X(p) = V_T(p) - V_0(p), \quad (2.6)$$

where $V_T(p) = p_0 + \sum_{i=1}^N p_i V_{i,T}$ is the random MtM value of the portfolio p at time T and $V_0(p)$ is the initial fair portfolio value. This assumption is simplistic, if taken literally, because it is difficult to fit loss specifications other than those based on FVTPL into this setting. For example, the losses of a bank's loan portfolio are commonly specified as the actuarial "cash" credit losses over the time interval $(0, T)$ and not the change in MtM values of the loans. However, there are some good reasons that justify using the fair value approach for the computation of the EC as discussed in Klaassen and van Eeghen (2009) on p. 34. We could adjust our formalism to handle a mix of valuation principles but it would come at the cost of increasing the notational burden without adding much insight.¹⁶ Using Equation 2.6 simplifies the exposition.

With the P&L defined as in Equation 2.6, it is straightforward to incorporate our idea of liquidity risk into the EC and RAROC machinery. At time T liquidity problems in the form of liquidity calls might realize and the bank is forced to liquidate part of its asset portfolio.¹⁷ This extension leads naturally to random optimal liquidity costs at time T , which we denote by $C_T^\alpha(p)$. The idea is to subtract, per scenario $\omega \in \Omega$, the optimal liquidity costs from the standard P&L at T :

$$X^\alpha(p) := V_T(p) - V_0(p) - C_T^\alpha(p) = V_T^\alpha(p) - V_0(p) = X(p) - C_T^\alpha(p).$$

Notice that in a scenario in which the bank cannot meet the liquidity call and hence defaults (Type 2 liquidity risk), we assume that it incurs a loss equal to its full initial asset portfolio value: $p \notin \mathcal{L}^\alpha(\omega) \implies X^\alpha(\omega) = -V_0(p)$. Essentially this means that we view $-V_0(p)$ as an upper bound for the capital losses that can occur over $(0, T)$. This upper bound seems natural in the setting of a bank's EC modeling but in cases this is not applicable one would need to adjust the treatment of illiquid states.

Formally, the liquidity adjustment requires the specification of the liquidity call as a nonnegative random variable, i.e., $\alpha : \Omega \rightarrow \mathbb{R}_+$. In addition, we also need the random portfolio proceed function G_T or in fact N random asset proceed functions, each taking values in \mathcal{G} , i.e., the measurable functions $G_{i,T} : \Omega \rightarrow \mathcal{G}$ for all $i = 0, 1, \dots, N$. Alternatively, we could directly estimate the liquidity cost profile per scenario (Definition 2.11) to

¹⁶The idea would be to distinguish between a bank's whole asset portfolio and a sub-portfolio of it that is "available-for-liquidation", and adjust the definition of the P&L of the whole portfolio accordingly.

¹⁷Here again it is useful to think of the liquidity call as a cumulative net cash outflow, rather than an instant cash need.

determine the optimal liquidity costs.¹⁸

We define the liquidity-adjusted EC as the outcome of a monetary risk measure applied to the liquidity-adjusted P&L.¹⁹

Definition 2.16 Liquidity-adjusted EC. Given a monetary risk measure ρ , the *liquidity-adjusted economic capital* (L-EC) is a map $\text{L-EC} : \mathcal{P} \rightarrow \mathbb{R}$ given by $\text{L-EC}(p) := \rho(X^\alpha(p))$.

For completeness, the *Liquidity-adjusted Value at Risk* (L-VaR) of a portfolio is given by

$$\text{L-VaR}_\beta(p) := \text{VaR}_\beta(X^\alpha(p)) = \inf\{c \in \mathbb{R} \mid \mathbb{P}\{-X^\alpha(p) \leq c\} \geq \beta\}, \quad (2.7)$$

where $\beta \in (0, 1)$ is the confidence level, usually close to 1. The liquidity-adjusted analogue to *Expected Shortfall* of a portfolio, is given by

$$\text{L-ES}_\beta(p) := \text{ES}_\beta(X^\alpha(p)) = \frac{1}{1-\beta} \int_\beta^1 \text{L-VaR}_u(p) du, \quad (2.8)$$

for $X^\alpha(p) \in \mathcal{H}$. We can proceed in the obvious way to adjust RAROC as well.

Definition 2.17 Liquidity-adjusted RAROC. Given a liquidity-adjusted economic capital, the *liquidity-adjusted RAROC* (L-RAROC) is a map $\text{L-RAROC} : \mathcal{P} \rightarrow \mathbb{R}$ given by $\text{L-RAROC}(p) := \frac{E[X^\alpha(p)]}{\text{L-EC}(p)} = \frac{E[X(p)] - E[C_T^\alpha(p)]}{\text{L-EC}(p)}$.

Notice the double penalty due to illiquidity: a decrease of the numerator as well as an increase in the denominator. Also note that in case the EC model assigns positive probability to default by illiquidity states, L-RAROC takes these states into account at least in the numerator due to the expectation operator.

2.4.4 Liquidity-adjusted EC and capital determination

The question comes up whether the liquidity-adjusted EC of a portfolio can be interpreted as the minimum capital requirement of a bank, i.e., is a solution to the earlier discussed capital determination problem. Interestingly, Anderson et al. (2010) argue against the use of $\rho(X^\alpha(p))$ because it fails to be “cash-invariant” or in our terminology cash-equity translationally anti-variant (see later results Theorem 2.26). For that reason, it fails according to them to meet the minimum capital requirement interpretation. They offer an alternative definition which equates the riskiness of a portfolio to the minimum amount of cash that needs to be added to the initial portfolio to make the risk at time T zero.²⁰ By doing this, their liquidity-adjusted risk measure is “cash-invariant” by construction. However, we disagree with their reasoning because we clearly retain the minimum capital requirement interpretation for liquidity-adjusted EC as long as

¹⁸Note that we assume that proceed functions are completely observable in a scenario at T . Clearly, this assumption is simplistic because in reality we would not have such a complete knowledge at any point in time.

¹⁹Acerbi and Scandolo (2008) consider $\rho(V_T^\alpha(p))$ instead. As we deal with monetary risk measures this difference is negligible.

²⁰More formally, Anderson et al. (2010) define the risk of a portfolio, using a slight abuse of notation, as $\tilde{\rho}(X^\alpha(p)) := \inf\{m \in \mathbb{R} \mid \rho(X^\alpha(p+m)) \leq 0\}$.

the underlying risk measure ρ is a monetary risk measure on the value level, which L-EC is by definition. To see this, notice that Equation 2.4 simply becomes

$$\rho(X^\alpha(p)) = \inf\{m \in \mathbb{R} \mid \rho(X^\alpha(p) + m) \leq 0\}, \quad (2.9)$$

which shows that the approach of Anderson et al. (2010) is not needed to “resolve the conceptual deficiencies” related to the minimum capital requirement interpretation with liquidity risk. In fact, we believe that with their definition of liquidity-adjusted risk measures they lose the link to practice and the capital requirement interpretation. Of course, the definition used in Anderson et al. (2010) is not meaningless, but they effectively consider a capital budgeting problem and not the bank’s capital determination problem anymore. Under the capital budgeting perspective we take the risk of a portfolio as the minimum quantity invested in any marketable security such that the original portfolio, along with the modified security, becomes acceptable (cf., Jarrow and Purnanandam (2005)). Within this context, Anderson et al. (2010) consider the special case of investing purely into cash. However, following Jarrow and Purnanandam (2005) this approach is not the supervisory perspective and the capital determination problem but rather a firm’s capital budgeting perspective. This confusion might be explained by the misleading use of the terms “cash” and “capital” in the risk measure theory literature. The two terms are often used interchangeably, which is rather harmless in the standard setting due to the linearity of MtM valuation function but causes problems in a liquidity risk formalism where the portfolio value is not a linear function on the vector space of portfolios anymore. As a result, focusing on “cash-invariance” in the liquidity risk formalism, like Anderson et al. (2010) do, leads to something different than the capital determination perspective.

However, there is an actual but different problem with the capital buffer interpretation in our framework despite the fact that Equation 2.9 holds in general. While it is true that Equation 2.9 must hold to speak of the minimum capital requirement interpretation of $\rho(X^\alpha(p))$, it is also necessary that we can interpret, allowing for some abstractions, $X^\alpha(p)$ as capital gains and losses in all scenarios. Unfortunately, this does not hold true in our liquidity risk formalism due to the way we operationalize Type 2 liquidity risk. Recall, that we formalize two types of liquidity risk:

1. Type 1 liquidity risk: scenarios where there is a positive liquidity call at time T and the bank can generate enough cash to meet it, incurring some nonnegative liquidity costs that deflate the standard P&L at time: $X(\omega) - C_T^A(\omega)$ with $C_T^A(\omega) < V_0(p)$.
2. Type 2 liquidity risk: scenarios where there is a positive liquidity call at time T and the bank cannot generate enough cash to meet it, hence defaults, incurring a 100% value loss: $X(\omega) - C_T^A(\omega) = -V_0(p)$.

While we technically assign a capital loss of $-V_0(p)$ to scenarios in which the bank cannot service its liquidity call, it cannot really be interpreted as a capital loss, i.e., it is just a formal means to treat default states. The problem is best illustrated by a simple example.

Example 2.18 Capital requirement interpretation. Consider we have two portfolio pairs \hat{p} and \hat{q} with the same asset portfolio but different funding structures, so that $A(\hat{p}) < A(\hat{q})$. The initial portfolio value is $V_0(p) = 190$. Consider the following three-state example:

	$X(p)$	$X^A(\hat{p})$	$X^A(\hat{q})$
$\omega_1, \mathbb{P}\{\omega_1\} = 0.95$	20	20	20
$\omega_2, \mathbb{P}\{\omega_2\} = 0.04$	-5	-5	-5
$\omega_3, \mathbb{P}\{\omega_3\} = 0.01$	-25	-50	-190 [†]
$\text{VaR}_{0.05}$	5	5	5
$\text{ES}_{0.05}$	9	14	42

[†] Default by illiquidity state, i.e., loss equals $-V_0(p)$.

There are two important things we can observe from the example. The first thing we notice is that

$$\text{L-VaR}_{0.05}(\hat{p}) = \text{L-VaR}_{0.05}(\hat{q}) = \text{VaR}_{0.05}(p),$$

despite the presence of liquidity risk and the occurrence of default by illiquidity states for portfolio pair \hat{q} but not for \hat{p} . Both forms of insensitivity do not occur when we consider the expected shortfall, as we have that

$$\text{L-ES}_{0.05}(\hat{q}) > \text{L-ES}_{0.05}(\hat{p}) > \text{ES}_{0.05}(p).$$

Secondly, consider the following two differences:

$$\Delta_1 := \text{L-ES}_{0.05}(\hat{p}) - \text{ES}_{0.05}(p) = 5 \text{ and } \Delta_2 := \text{L-ES}_{0.05}(\hat{q}) - \text{ES}_{0.05}(p) = 33.$$

The first difference Δ_1 can readily be interpreted and justified as an increase in capital requirement due to liquidity risk because the increase is purely caused by Type 1 liquidity risk. However, the justification of the second difference Δ_2 as a necessary increase in capital requirements is more problematic since it is purely driven by Type 2 liquidity risk and the 100% rule. The minimum amount of capital needed to cover insolvency risk is 9 but now because of the possibility of default by illiquidity in the third scenario (ω_3) we ask for capital of 42. While Equation 2.9 still applies, it is difficult to assign a real world meaning to the number 42 because the 100% rule is factored in. ■

Where does that lead us? First of all it is useful to realize that the insensitivity of L-VaR in the previous example is not an accident. It is easily verified that L-VaR_β is insensitive to the default by illiquidity states whenever

$$\beta > \text{PD}^{\text{liq}} := \mathbb{P}\{\omega \in \Omega \mid \hat{p} \notin \mathcal{L}^\alpha(\omega)\}, \quad (2.10)$$

due to the 100% rule. On this matter there is a sharp difference between L-VaR and L-ES, or in fact *spectral risk measures* of which L-ES is a special case. Whenever we use a spectral risk measures as the underlying risk measure, we cannot avoid the fact that illiquid states are taken into account because of the monotonicity of the

risk-aversion function (see Proposition 3.4 p. 171 in Acerbi, 2004). That means that whenever the underlying risk measure of L-EC is sensitive to default states by illiquidity given a probability measure, interpreting the outcome purely as capital requirement is problematic, despite the fact that Equation 2.9 holds in general.

From a conceptual point of view one can argue that for the above reasons Type 1 liquidity risk but not Type 2 liquidity risk should be included in EC computations. For instance, Klaassen and van Eeghen (2009) come to such a conclusion on page 44:

We conclude that market liquidity risk in the form of price risk of trading assets [Type 1 liquidity risk] and liabilities, or in the form of higher funding costs than expected, should be included in economic capital. Funding liquidity risk does not correspond to a potential increase in funding costs, but to the unavailability of funding [Type 2 liquidity risk]. We have argued that it should not be included in economic capital because it does not represent a cause for a decline in capital, but rather can be a consequence of such a decline.

While we agree with their basic line of reasoning, we also believe that penalizing banks not only for Type 1 liquidity risk but also for Type 2 liquidity risk *via* capital requirements (EC) outweighs this conceptual problem. We believe that because capital requirements and EC play such a prominent role as a management control tool within banks and as a signaling tool in the financial world, it would be advantageous to incorporate as many aspects of liquidity risk into it. Also note that an alternative approach, as suggested by Brunnermeier et al. (2009), to include liquidity risk into capital requirements via a multiplier applied to the standard capital charge based on the effective mismatch between the asset maturity and the liability maturity, is far more detached from the conceptual idea of the capital determination problem.

Apart from conceptual issues of including liquidity risk into capital requirements, we also have to deal with the practical problem of probabilistic modeling liquidity risk. As mentioned earlier, there is often the sentiment that “it is difficult to quantify” liquidity risk and hence it should be analyzed predominately with the help of stress scenarios and not full probability models (p. 6 in BIS (2009)). It is clear that for the application of our formalism we need probability models for the liquidity risk components and we believe that there is no fundamental problem to achieve that. We discuss this issue in a bit more detail later in this chapter, as well as in Chapter 3.

Remark 2.4.1. We should stress that it is not the intention of our treatment of default by illiquidity and more general our L-EC to be an estimate in any sense of the capital levels that is needed to prevent funding problems to materialize in any or some scenario in the first place. It is merely meant as a reasonable treatment of a bank's liquidity risk.

Remark 2.4.2. In practice, the bank's probability of default (PD) implied by the EC model is relevant for the determination of the bank's credit rating. It is straightforward to see that in our formalism with liquidity risk there is no change in the intuition. The implied PD in the standard setting is given by $PD_X := \mathbb{P}\{\omega \in \Omega \mid -X(\omega) \geq \ell_0\}$, where ℓ_0

is the actual loss-absorbing capital of the bank. Using the 100% rule we are ensured that in our formalism the implied PD includes default by insolvency and default by illiquidity: $\text{PD}_{X^\alpha} := \mathbb{P}\{\omega \in \Omega \mid -X^\alpha(\omega) \geq \ell_0\}$. It is also easily verified that $\text{PD}_{X^\alpha} \geq \text{PD}_X$.

Remark 2.4.3. Another way to look at our treatment of default by illiquidity states is that we exclude, via the 100% rule, scenarios in which a bank can be solvent and illiquid at the same time, i.e., scenarios of the form $X(\omega) > -\ell_0$ while $\hat{p} \notin \mathcal{L}^A(\omega)$.

2.5 Diversification, coherent risk measures, and liquidity risk

In this section, we extend our formalism by asset and liability pairings and liquidity call functions with the purpose to study the properties of liquidity-adjusted risk measures under basic portfolio manipulations. In particular, we study the effect of liquidity risk on the diversification principle of risk. The main results in this section might help banks interested in applying the idea of L-EC to choose an appropriate underlying risk measure. In addition, our results should dispel concern about the validity of the coherency axioms in the presence of liquidity risk.

2.5.1 Portfolio pairs and liquidity call functions

Suppose that the bank's asset portfolio is funded by a *liability portfolio*. We assume that there are $M + 1$ liabilities or liability classes available in the market, indexed by $j = 0, 1, \dots, M$, with $j = 0$ being reserved for equity. Essentially we now assume that a bank's exposure is represented by a *portfolio pair* and not just an asset portfolio.

Definition 2.19 Portfolio pair. A *portfolio pair* \hat{p} is a nonnegative real-valued pair of vectors consisting of an asset portfolio and a liability portfolio, $\hat{p} = (p, \ell) \in \hat{\mathcal{P}} = \mathbb{R}_+^{N+1} \times \mathbb{R}_+^{M+1}$.

From now on, operations on portfolios must always be specified in terms of pairs. While formally trivial, we believe it is economically significant because it forces us to think in terms of asset/liability pairs and the additional risk posed by the funding structure. In order to link liability portfolios with funding risk we introduce the concept of a liquidity call functions. This function maps a portfolio pairs \hat{p} to a random nonnegative liquidity call $\alpha \in \mathcal{H}$. We assume that cash funded by equity can be seen as risk-free in the liquidity risk context, in that it does not produce a liquidity call in any scenario and has no market liquidity risk. This makes sense as we interpret liquidity calls as sudden cash obligations in times of stress.

Definition 2.20 Liquidity call function. A *liquidity call function* A is a measurable function $A : \hat{\mathcal{P}} \rightarrow \mathcal{H}$, with $\alpha \in \mathcal{H}$ a random liquidity call related to \hat{p} and we have that $A(0) = 0$ and

ETI Equity-cash translation-invariance:

$$\text{for all } \hat{p} \in \hat{\mathcal{P}} \text{ and for all } \hat{q} \in \hat{\mathcal{P}}_{0,0}, A(\hat{p} + \hat{q}) = A(\hat{p}).$$

MON Monotonicity:

$$\text{for all } \hat{p} \geq \hat{q}, A(\hat{p}) \geq A(\hat{q}).$$

We denote the space of liquidity call functions by \mathcal{A} .

Note that A is a function of \hat{p} and not just ℓ . While we associate liquidity calls predominately with the liabilities of a bank, the general formulations allows us to consider all forms of liquidity calls. For our main results we will consider some suitable subsets of \mathcal{A} . An important subset is the set of liquidity call functions that are *convex* per scenario $\omega \in \Omega$:

$$\text{for all } \hat{p}, \hat{q} \in \hat{\mathcal{P}} \text{ and for all } \theta \in [0, 1] \quad A(\theta \hat{p} + (1 - \theta)\hat{q}) \leq \theta A(\hat{p}) + (1 - \theta)A(\hat{q}).$$

Clearly convexity of A does not give justice to all subtleties involved in funding liquidity risk, but at least it captures the intuition that diversification is beneficial, which might be justified by mutually mitigating effects in corresponding cash flow schemes and diversification effects regarding the funding according to investor type, geography, instruments, currency and so on.²¹ Note that convexity of A implies (cf. Lemma 2.6) that for all $\hat{p} \in \hat{\mathcal{P}}$ and for all $\lambda \geq 1$, $A(\lambda \hat{p}) \geq \lambda A(\hat{p})$, as well as, for all $\hat{p} \in \hat{\mathcal{P}}$ and for all $\lambda \in [0, 1]$, $A(\lambda \hat{p}) \leq \lambda A(\hat{p})$.

Also noteworthy is the class of *subadditive* (and perhaps positively homogenous) A and the class of *linear* A . The assumption of subadditivity make sense, if one believes that adding portfolio pairs can only be beneficial in terms of funding liquidity risk. The class of linear liquidity call functions might be the most natural for applications in practice and takes the form of

$$A(\hat{p}) = \sum_{i=1}^N p_i \alpha_i + \sum_{j=1}^M \ell_j \alpha_j,$$

where the α 's can be interpreted as the liquidity calls at time T or we could view it as a cumulative cash need over $(0, T)$. Notice that the assumption of linear liquidity call functions would not automatically mean that we can avoid the nonlinear scale effects in *liquidity costs*.

Random MtM values, random portfolio proceeds, and random liquidity calls, lead to random optimal liquidity costs of a portfolio pair, which we denote by $C_T^A(\hat{p})$. In other words, we simply substitute α with $A(\hat{p})$. The liquidity-adjusted P&L under a liquidity call function $A \in \mathcal{A}$ is given by

$$X^A(\hat{p}) := V_T(p) - C_T^A(\hat{p}) - V_0(p) = V_T^A(\hat{p}) - V_0(p) = X(p) - C_T^A(\hat{p}).$$

Remark 2.5.1. Our liquidity call function is similar to the short-term cash flow function in Anderson et al. (2010), which they take to be a *non-positive* concave function.

²¹Most banks emphasize the importance of diversifying its funding as a means to manage liquidity risk (Matz and Neu, 2007).

2.5.2 Properties of optimal liquidity costs under liquidity call functions

As a preparation for our main results we characterize the optimal liquidity cost function under a liquidity call function. All results hold per scenario $\omega \in \Omega$.

Lemma 2.21. *If A is convex, the liquidity feasibility set given by $\mathcal{L}^A := \{\hat{p} \in \hat{\mathcal{P}} \mid G_T(p) \geq A(\hat{p})\}$ is convex and closed under downscaling but not necessarily closed under upscaling.²²*

Proof of Lemma 2.21.

- The liquidity acceptability set under a convex liquidity call function A , given by $\mathcal{L}^A = \{\hat{p} \in \hat{\mathcal{P}} \mid G_T(p) - A(\hat{p}) \geq 0\}$, is the upper level set of a concave function, which is convex for any level value (see, e.g., Proposition 2.7 in Rockafellar and Wets (2004)).
- For $\tau \in [0, 1]$ and $\hat{p} \in \mathcal{L}^A$ it is easily verified that $G_T(\tau \hat{p}) \geq \tau G_T(\hat{p}) \geq \tau A(\hat{p}) \geq A(\tau \hat{p})$ by the concavity of G_T , convexity of A , and the fact that both are zero in zero.
- Let us find a counterexample to prove the third claim. Consider some convex $A \in \mathcal{A}$ and a portfolio pair such that $G_T(p) = A(\hat{p})$ and for all $\tau > 1$, $G_T(\tau p) = G_T(p)$. Clearly, $\hat{p} \in \mathcal{L}^A$. Now consider upscaling \hat{p} by τ . In order for $\tau \hat{p}$ to be in \mathcal{L}^A , it must hold that $G_T(\tau p) \geq A(\tau \hat{p})$. However, it follows from Lemma 2.6 and positively super-homogeneity of A that $G_T(\tau p) = G_T(p) < \tau G_T(p) = \tau A(\hat{p}) \leq A(\tau \hat{p})$, which implies that $\tau \hat{p} \notin \mathcal{L}^A$. \square

Some of the results only hold for pairs of portfolio pairs that are either both liquid or both illiquid almost surely:

$$\mathcal{M} := \{(\hat{p}, \hat{q}) \in \hat{\mathcal{P}} \mid \forall \omega \in \Omega \quad \hat{p}, \hat{q} \in \mathcal{L}^A(\omega) \text{ or } \hat{p}, \hat{q} \notin \mathcal{L}^A(\omega)\}. \quad (2.11)$$

We summarize the basic properties of the optimal liquidity costs in the next theorem.

Theorem 2.22.

- CONV C_T^A is convex on \mathcal{M} if A is convex.
- SUB I C_T^A is subadditive on \mathcal{M} if A is subadditive and \hat{p} and \hat{q} have no non-cash assets in common: for all $(\hat{p}, \hat{q}) \in \mathcal{M}$ such that $p_i q_i = 0$ for all $i > 0$, $C_T^A(\hat{p} + \hat{q}) \leq C_T^A(\hat{p}) + C_T^A(\hat{q})$
- SUB II C_T^A is subadditive on \mathcal{L}^A for portfolios pairs that do not demand liquidity calls if A is convex: for all $\hat{p} \in \mathcal{L}^A$ and all $\hat{q} \in \hat{\mathcal{P}}$ such that $A(\hat{q}) = 0$, $C_T^A(\hat{p} + \hat{q}) \leq C_T^A(\hat{p})$
- SUP C_T^A is superadditive on \mathcal{M} if A is linear and if the marginal liquidity costs coincide:
for all $(\hat{p}, \hat{q}) \in \mathcal{M}$ such that $\lambda_{p,T} = \lambda_{q,T}$, $C_T^A(\hat{p} + \hat{q}) \geq C_T^A(\hat{p}) + C_T^A(\hat{q})$
- PSUPH C_T^A is positively super-homogenous if A is convex: for all $\hat{p} \in \hat{\mathcal{P}}$ and all $\lambda \geq 1$, $C_T^A(\lambda \hat{p}) \geq \lambda C_T^A(\hat{p})$

²²It is easily verified that $\mathcal{P} \setminus \mathcal{L}^A$ is closed under upscaling.

Proof of Theorem 2.22.

CONV Because the claim is restricted to \mathcal{M} we have to consider two cases: (a) $\hat{p}, \hat{q} \in \mathcal{L}^A$ and (b) $\hat{p}, \hat{q} \notin \mathcal{L}^A$. Let $\hat{p}, \hat{q} \in \mathcal{L}^A$ and $\hat{p}_\theta := \theta \hat{p} + (1 - \theta) \hat{q}$ for $\theta \in [0, 1]$. From Lemma 2.21 we know that $\hat{p}_\theta \in \mathcal{L}^A$. Furthermore, let x_1^* , x_2^* , and $x_{\hat{p}_\theta}^*$ be the optimal liquidation strategies associated with $C_T^A(\hat{p})$, $C_T^A(\hat{q})$, and $C_T^A(\hat{p}_\theta)$, respectively. Defining $x_\theta^* := \theta x_1^* + (1 - \theta) x_2^*$ and using the concavity of G and the convexity of A , it follows that $G_T(x_\theta^*) = G_T(\theta x_1^* + (1 - \theta) x_2^*) \geq \theta G_T(x_1^*) + (1 - \theta) G_T(x_2^*) = \theta A(\hat{p}) + (1 - \theta) A(\hat{q}) \geq A(\hat{p}_\theta) = G_T(x_{\hat{p}_\theta}^*)$. So x_θ^* is a liquidation strategy that generates sufficient cash ($\geq A(\hat{p}_\theta)$) at costs below $\theta C_T^A(\hat{p}) + (1 - \theta) C_T^A(\hat{q})$ (by convexity). The result follows.

For the second case, \hat{p} and $\hat{q} \notin \mathcal{L}^A$, we need to consider two possible situations: (1) $\hat{p}_\theta \notin \mathcal{L}^A$ and (2) $\hat{p}_\theta \in \mathcal{L}^A$. Both situations are trivial. In situation (1) we have equality by linearity of V and in situation (2) we have that $C_T^A(\hat{p}_\theta) \leq V_T(\theta p + (1 - \theta) q)$, which always holds as $C_T^A(\hat{p}_\theta) \in [0, V_T(\theta p + (1 - \theta) q)]$ by definition.

SUB I Again we have to consider two cases: (a) $\hat{p}, \hat{q} \in \mathcal{L}^A$ and (b) $\hat{p}, \hat{q} \notin \mathcal{L}^A$. Suppose x^* and s^* denote the optimal liquidation strategy for $\hat{p}, \hat{q} \in \mathcal{L}^A$. Given that \hat{p} and \hat{q} have no non-cash assets in common, it follows that $G_T(x^* + s^*) = G_T(x^*) + G_T(s^*) = A(\hat{p}) + A(\hat{q})$ by definition of G . As we assume that A is subadditive, it follows that the liquidation strategy $x^* + s^*$ will always generate at least the liquidity call $A(\hat{p} + \hat{q})$, hence $\hat{p} + \hat{q} \in \mathcal{L}^A$, and the result follows.

For the second case, $\hat{p}, \hat{q} \notin \mathcal{L}^A$, we need to consider two possible situations: (1) $\hat{p} + \hat{q} \notin \mathcal{L}^A$ and (2) $\hat{p} + \hat{q} \in \mathcal{L}^A$. Both situations are trivial. In situation (1) we have equality by linearity of V and in situation (2) we have that $C_T^A(\hat{p} + \hat{q}) \leq V_T(p + q)$, which always holds as $C_T^A(\hat{p} + \hat{q}) \in [0, V_T(p + q)]$ by definition.

SUB II Given that $\hat{p} \in \hat{\mathcal{P}}$ and $\hat{q} \in \hat{\mathcal{P}}$ such that $A(\hat{q}) = 0$, we have that $A(\hat{p} + \hat{q}) = A(\hat{p})$, hence $C_T^A(\hat{q}) = 0$. It is easily verified that adding \hat{q} to \hat{p} can never increase the optimal liquidity costs.

SUP Given that the domain is restricted to \mathcal{M} , we need to consider three cases: (a) $\hat{p}, \hat{q}, \hat{p} + \hat{q} \in \mathcal{L}^A$, (b) $\hat{p}, \hat{q} \in \mathcal{L}^A, \hat{p} + \hat{q} \notin \mathcal{L}^A$, and (c) $\hat{p}, \hat{q}, \hat{p} + \hat{q} \notin \mathcal{L}^A$. Note that it is easily verified that the case $\hat{p}, \hat{q} \notin \mathcal{L}^A, \hat{p} + \hat{q} \in \mathcal{L}^A$ cannot occur due to linearity of A and subadditivity of G . Case (b) and (c) are trivial. For case (b) the claim reduces to $V_T(p + q) \geq C_T^A(\hat{p}) + C_T^A(\hat{q})$ which is always true in general by definition. For case (c) the claim reduces to $V_T(p + q) \geq V_T(p) + V_T(q)$ which holds of course with equality by the linearity of V .

For case (a) suppose that r^* , s^* , and \bar{r}^* denote the optimal liquidation strategy for $\hat{p}, \hat{q}, \hat{p} + \hat{q} \in \mathcal{L}^A$, generating $G_T(r^*) = A(\hat{p})$, $G_T(s^*) = A(\hat{q})$ and by

linearity of A , $G_T(\bar{r}^*) = A(\hat{p}) + A(\hat{q})$ cash, respectively. Then

$$\begin{aligned}
C_T^A(\hat{p} + \hat{q}) &= V_T(\bar{r}^*) - G_T(\bar{r}^*) \\
&= V_T(r^*) - G_T(r^*) + V_T(s^*) - G_T(s^*) + V_T(\bar{r}^* - r^* - s^*) \\
&\quad - G_T(\bar{r}^*) + G_T(r^*) + G_T(s^*) \\
&= C_T^A(\hat{p}) + C_T^A(\hat{q}) + V_T(\bar{r}^* - r^* - s^*) - G_T(\bar{r}^*) + G_T(r^*) + G_T(s^*) \\
&= C_T^A(\hat{p}) + C_T^A(\hat{q}) + V_T(\bar{r}^* - r^* - s^*),
\end{aligned}$$

where we use in the last step that $G_T(\bar{r}^*) = A(\hat{p} + \hat{q}) = G_T(r^*) + G_T(s^*) = A(\hat{p}) + A(\hat{q})$. Hence, we need to show that $V_T(\bar{r}^* - r^* - s^*) = V_T(\bar{r}^*) - V_T(r^* + s^*) \geq 0$ for $\lambda_{p,T} = \lambda_{q,T}$. Because we have that $0 \leq \bar{r}^* \leq p + q$, we can find a decomposition $\bar{r}^* = \bar{r}_p^* + \bar{r}_q^*$ such that $0 \leq \bar{r}_p^* \leq p$ and $0 \leq \bar{r}_q^* \leq q$. This decomposition need not be unique. By subadditivity of G (Lemma 2.6) we have that $G_T(\bar{r}_p^*) + G_T(\bar{r}_q^*) \geq G_T(\bar{r}_p^* + \bar{r}_q^*) = G_T(r^*) + G_T(s^*) = A(\hat{p}) + A(\hat{q})$. So either (i) $G_T(\bar{r}_p^*) \geq A(\hat{p})$ or (ii) $G_T(\bar{r}_q^*) \geq A(\hat{q})$. In case (i), the other case is entirely similar, define $\delta := G_T(\bar{r}_p^*) - A(\hat{p})$. Recall that by definition of the marginal liquidity proceeds, the extra amount δ is liquidated at reduction in MtM value of at least $\delta \mu_{p,T}$ with $\mu_{p,T} = 1/(1 - \lambda_{p,T})$. Furthermore, $G_T(\bar{r}_q^*) \geq G_T(\bar{r}_p^* + \bar{r}_q^*) - G_T(\bar{r}_p^*) = A(\hat{q}) - \delta$, so \bar{r}_q^* generates at most δ cash less than s^* , and again by definition of marginal liquidity proceeds the reduction in consumed MtM-value is at most $\delta \mu_{q,T}$ with $\mu_{q,T} = 1/(1 - \lambda_{q,T})$. Clearly, if $\mu_{p,T} = \mu_{q,T}$, the net effect on consumed MtM-value is nonnegative, i.e., $V_T(\bar{r}^* - r^* - s^*) \geq 0$.

PSUPH We have to consider three cases: (1) $\hat{p}, \lambda \hat{p} \in \mathcal{L}^A$, (2) $\hat{p} \in \mathcal{L}^A, \lambda \hat{p} \notin \mathcal{L}^A$ and (3) $\hat{p} \notin \mathcal{L}^A$. For the first case, we refer the reader to Lemma 2.6. For the second case, PSUPH follows easily because $C_T^A(\lambda \hat{p}) = V_T(p) \geq \lambda C_T^A(\hat{p})$. The third case is straightforward as well, as it follows from Lemma 2.21 that $C_T^A(\lambda \hat{p}) = \lambda C_T^A(\hat{p}) = \lambda V_T(p)$. \square

Convexity shows that blending portfolio pairs has a positive *diversification effect* on liquidity costs if the liquidity call function is convex. However, this positive effect only occurs if we exclude certain distorting effects of Type 2 liquidity risk via \mathcal{M} . Positive super-homogeneity of liquidity costs shows that doubling a portfolio pair generally leads to more than twice the liquidity costs per scenario. This negative upscaling effect can mostly be motivated by limited market liquidity. General additivity results in terms of portfolio pairs cannot be made. There are two opposing effects of liquidity risk with regard to the liquidation problem: (1) a positive *diversification effect* of liquidity sources, and (2) a negative *concentration effect* of liquidity sources. The diversification effect says that portfolios pairs may benefit from extra sources of liquidity if another portfolio pair is added. In that case, extra liquidity sources have a sparing effect that allows the bank to avoid liquidating less liquid sources at higher costs. SUB I and SUB II represent cases where the diversification effect dominates. The intuition behind SUB I and SUB II is straightforward. In the case of SUB I we cannot suffer from the concentration

effect of liquidity sources (cf., Definition 2.4). However, note that for SUB I we need to assume that A is subadditive to rule out that the diversification effect is overpowered by increased liquidity calls. In the case of SUB II funding risk does not increase and hence adding extra portfolio pairs is never harmful and often beneficial. In contrast, the negative concentration effect says that common assets (or asset categories) in both portfolio pairs may lead to liquidation at higher volumes in those assets and hence higher overall liquidity costs due to the subadditivity of the portfolio proceed functions (see Lemma 2.6). This element is most prominent if one considers upscaling (PSUPH), but also for combining portfolio pairs with the same marginal liquidity costs. Notice that the latter result holds only under linear liquidity call functions because subadditive A could overpower the concentration effect.

Remark 2.5.2. Adding cash-equity portfolio pairs to a portfolio pair is a special case of SUB II for which we can find upper and lower bounds for the decrease in the optimal liquidity costs per scenario. Let the set of all cash-equity portfolio pairs be given by

$$\widehat{\mathcal{P}}_{0,0} := \{\widehat{p} \in \widehat{\mathcal{P}} \mid p_0 = \ell_0, p_i = 0 \text{ for } i = 1, \dots, N, \ell_j = 0 \text{ for } j = 1, \dots, M\}.$$

Then for all $\widehat{p} \in \mathcal{L}^A$ and for all $\widehat{q} \in \widehat{\mathcal{P}}_{0,0}$ it holds that $q_0 \lambda_p(A(\widehat{p}) - q_0) \leq C_T^A(\widehat{p}) - C_T^A(\widehat{p} + \widehat{q}) \leq q_0 \lambda_{p,T}(A(\widehat{p}))$ per scenario. The bound is easily derived as soon as we notice that the cost reduction is given by $C_T^A(\widehat{p}) - C_T^A(\widehat{p} + \widehat{q}) = \int_{A(\widehat{p}) - q_0}^{A(\widehat{p})} \lambda_p(s) ds$.

We state the properties of the closely related liquidity-adjusted value as well as the liquidity-adjusted P&L in the following corollary, which follows Theorem 2.22.

Corollary 2.23.

CONC V_T^A and X^A are concave on \mathcal{M} if A is convex.

SUP I V_T^A and X^A is superadditive on \mathcal{M} if A is subadditive and \widehat{p} and \widehat{q} have no non-cash assets in common

SUP II V_T^A and X^A is superadditive on \mathcal{L}^A for portfolios pairs that do not demand liquidity calls if A is convex

SUB V_T^A and X^A is subadditive on \mathcal{M} if A is linear and if the marginal liquidity costs coincide

PSUBH V_T^A and X^A is positively sub-homogenous if A is convex

Note that if we write SUP II out we get $V_T^A(\widehat{p} + \widehat{q}) \geq V_T^A(\widehat{p}) + V_T(q)$, which tells us that the adding a portfolio pair with no funding risk increases the liquidity-adjusted value of the combined asset portfolio by more than the MtM value of the added asset portfolio per scenario.

2.5.3 Coherent risk measures and liquidity risk

A popular class of risk measures are coherent risk measures (Artzner et al., 1999) as the axioms defining the class are widely accepted as being useful and convincing in the financial context.

Definition 2.24 Coherent risk measure. A monetary risk measure $\rho : \mathcal{H} \rightarrow \mathbb{R}$ is called a *coherent risk measure* if it satisfies the following properties:

- PH Pos. homogeneity of degree one:
for all $X \in \mathcal{H}$ and all $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$
- SUB Subadditivity:²³
for all $X, Y \in \mathcal{H}$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$

The coherency axioms produced two main discussions. Firstly, the original authors showed that VaR, the most widely used risk measure in the financial world, is not coherent as it fails to be subadditive in general, which violates the intuitive notion of the diversification principle.²⁴ Secondly and more important to our setting, some academics argue that the axioms PH and S clash with our intuition regarding liquidity risk. The brunt of the criticism can be summarized by the following proposition:

If one doubles an illiquid asset portfolio, the risk becomes more than double as much, but according to the PH axiom this is not allowed!

Some suggest to generalize the axioms itself (Foellmer and Schied, 2002; Heath and Ku, 2004; Frittelli and Gianin, 2002) by replacing the axioms on subadditivity and positive homogeneity by the weaker convexity,²⁵ while others (Jarrow and Protter, 2005) retain the coherent axioms but argued for a liquidity cost adjustment on the value level. We follow the spirit of the latter approach as we agree with Acerbi and Scandolo (2008)'s defense of the coherency axioms. The authors rightfully point out that the liquidity risk argument in favor of loosening the axioms is essentially based on a *category mistake* involving a confusion of values, represented by $X \in \mathcal{H}$, and positions, in our notation given by $p \in \mathcal{P}$. The axioms of coherent risk measures can only be interpreted in terms of values (by AM) as they were always meant to be,²⁶ but the proposition is implicitly talking about units/positions and not values. Of course, in the linear case we have, by positive homogeneity of ρ , that

$$\rho(X(2p)) = \rho(2X(p)) = 2\rho(X(p)),$$

so that doubling a portfolio leads to a doubling of the P&L and hence a doubling of the riskiness. But as soon as we move to a nonlinear exposure relationship, such as we have

²³Throughout, we assume that the P&L X and Y as random variables are not affected by whether or not the positions are merged. We do as if only the legal liability construction changes. In reality, merging or splitting positions / balance sheets may change management, business strategy, cost structure, etc., and may thus change the P&L under consideration.

²⁴We refer the interested reader to Example 6.7 on p. 241 in McNeil et al. (2005) for an illustration of what can go wrong when using VaR in some cases.

²⁵A monetary risk measure $\rho : \mathcal{H} \rightarrow \mathbb{R}$ is called a *convex risk measure* if for all $X, Y \in \mathcal{H}$ and for all $\lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$. It is easily verified that the space of coherent risk measures is a proper subset of the space of convex risk measures.

²⁶In Artzner et al. (1999) we read: "If position size directly influences risk (for example, if positions are large enough that the time required to liquidate them depend on their sizes) then we should consider the consequences of lack of liquidity when computing the future net worth [P&L] of a *position*. With this in mind, Axioms S and PH about mappings from *random variables* into the reals, remain reasonable."

in our formalism, this result breaks down in general:

$$\rho(X^\alpha(2p)) \neq \rho(2X^\alpha(p)) = 2\rho(X^\alpha(p)).$$

However, note that the right hand side still applies and still makes sense.

We conclude that the coherency axioms are not in conflict with basic intuition about liquidity risk because of the distinction between positions and values. In the next section, we look into the properties of liquidity-adjusted risk measures under some basic portfolio manipulations such as addition, blending, and scaling.

2.5.4 Properties of liquidity-adjusted risk measures

Given that we extended our setting by portfolio pairs and liquidity call functions, we adjust our previous definitions of liquidity-adjusted P&L slightly. In particular, we pull the actual initial equity position of the bank, denoted by ℓ_0 , into the definition, assuming that before the realization of X^A , the capital of the bank at time T is equal to the initial capital:

$$\bar{X}^A(\hat{p}) := X^A(\hat{p}) + \ell_0.$$

This way ℓ_0 acts as a loss buffer, which is in line with the capital adequacy and determination perspective. For this reason ℓ_0 should represent the available loss-absorbing capital of the bank today. Note that we assume that liabilities only affect the P&L of the bank indirectly via α and the corresponding liquidity cost term. In reality there clearly is a more direct impact due to interest rate / asset and liability management risk but we abstract from it here for simplicity and to isolate the acute liquidity risk effect.²⁷

Next we give the definition of liquidity-adjusted risk measures.

Definition 2.25 Liquidity-adjusted risk measure. Given a liquidity call function $A \in \mathcal{A}$ and a monetary risk measure ρ , a *liquidity-adjusted risk measure* is a map $\rho^A : \hat{\mathcal{P}} \rightarrow \mathbb{R}$ such that $\rho^A(\hat{p}) := \rho(\bar{X}^A(\hat{p})) = \rho(X^A(\hat{p})) - \ell_0$.

Of course the choice of the underlying ρ has important consequences in practice as well as in theory. While most academics prefer coherent risk measures over VaR because VaR is not subadditive in general, hence not coherent, VaR is used predominately in practice (see Remark 2.5.3). In the following theorem we summarize the properties of liquidity-adjusted risk measures under different assumptions regarding the underlying risk measure ρ and liquidity call function A .

Theorem 2.26. *Given a liquidity-adjusted risk measure ρ^A for some underlying monetary risk measure ρ and liquidity call function $A \in \mathcal{A}$. Then*

CONV ρ^A is convex on \mathcal{M} if both ρ and A are convex

²⁷Related to this issue is the risk of higher than expected funding costs due to general market disruptions or a decline in the credit rating of the bank. It is a form of market liquidity risk related to capital markets rather than secondary asset markets. However, some banks already take this dimension of liquidity risk into account via their interest rate risk treatment.

- PSUPH ρ^A is positively super-homogeneous if ρ is positively super-homogeneous and A is convex: for all $\hat{p} \in \hat{\mathcal{P}}$ and all $\lambda \geq 1$, $\rho^A(\lambda\hat{p}) \geq \lambda\rho^A(\hat{p})$
- SUB I ρ^A is subadditive on \mathcal{M} if both ρ and A are subadditive and if two portfolios pairs have no non-cash assets in common: for all $\hat{p}, \hat{q} \in \mathcal{M}$ such that $p_i q_i = 0$ for all $i > 0$, $\rho^A(\hat{p} + \hat{q}) \leq \rho^A(\hat{p}) + \rho^A(\hat{q})$
- SUB II ρ^A is subadditive on \mathcal{L}^A for portfolio pairs that do not demand liquidity calls if ρ is subadditive and A is convex: for all $\hat{p} \in \hat{\mathcal{P}}$ and all $\hat{q} \in \hat{\mathcal{P}}$ such that $A(\hat{q}) = 0$, $\rho^A(\hat{p} + \hat{q}) \leq \rho^A(\hat{p}) + \rho(X(q))$
- CTSUB ρ^A is cash-equity translationally subvariant on \mathcal{M} : for all $\hat{p} \in \mathcal{M}$ and $\hat{q} \in \hat{\mathcal{P}}_{0,0} \subseteq \mathcal{M}$, $\rho^A(\hat{p} + \hat{q}) \leq \rho^A(\hat{p}) - q_0$
- SUP ρ^A is superadditive on \mathcal{M} if ρ is additive, A is linear, and if the marginal liquidity costs coincide: for all $(\hat{p}, \hat{q}) \in \mathcal{M}$ such that $\lambda_{p,T} = \lambda_{q,T}$, $\rho^A(\hat{p} + \hat{q}) \geq \rho^A(\hat{p}) + \rho^A(\hat{q})$
- AM ρ^A is anti-monotone: for all $\hat{p}, \hat{q} \in \hat{\mathcal{P}}$ and $X^A(\hat{p}) \geq X^A(\hat{q})$, $\rho^A(\hat{p}) \leq \rho^A(\hat{q})$.

Proof of Theorem 2.26. All properties of C_T^A used in the proof can be found in Theorem 2.22.

- CONV With some minor rewriting and using TA of ρ we have that $\rho^A(\theta\hat{p} + (1-\theta)\hat{q}) = \rho(V_T^A(\theta\hat{p} + (1-\theta)\hat{q})) + \theta V_0(p) + (1-\theta)V_0(q) + c$, where $c = -\theta\ell_0^{\hat{p}} - (1-\theta)\ell_0^{\hat{q}}$. Proceeding similarly, we have that $\theta\rho^A(p) + (1-\theta)\rho^A(q) = \theta\rho(V_T^A(\hat{p})) + (1-\theta)\rho(V_T^A(\hat{q})) + \theta V_0(p) + (1-\theta)V_0(q) + c$. From concavity of V_T^A and from anti-monotonicity of ρ we know that $\rho(V_T^A(\theta\hat{p} + (1-\theta)\hat{q})) \leq \rho(\theta V_T^A(\hat{p}) + (1-\theta)V_T^A(\hat{q}))$. From convexity of ρ we have that $\rho(\theta V_T^A(\hat{p}) + (1-\theta)V_T^A(\hat{q})) \leq \theta\rho(V_T^A(\hat{p})) + (1-\theta)\rho(V_T^A(\hat{q})) = \theta\rho^A(p) + (1-\theta)\rho^A(q)$.
- PSUPH For $\lambda \geq 1$ and for all $\hat{p} \in \hat{\mathcal{P}}$ we have that $\rho^A(\lambda\hat{p}) = \rho(\bar{X}^A(\lambda\hat{p})) = \rho(\lambda X(p) - C_T^A(\lambda\hat{p}) + \lambda\ell_0)$. Furthermore, we have using positive super-homogeneity of ρ that $\rho(\lambda\bar{X}^A(\hat{p})) = \rho(\lambda X(p) - \lambda C_T^A(\hat{p}) + \lambda\ell_0) \geq \lambda\rho^A(\hat{p})$. The result follows directly from positive super-homogeneity of C_T^A (Theorem 2.22) and anti-monotonicity of ρ .
- SUB I From SUB I of C_T^A , anti-monotonicity, and subadditivity of ρ we have that $\rho^A(\hat{p} + \hat{q}) = \rho(\bar{X}^A(\hat{p} + \hat{q})) \leq \rho(\bar{X}^A(\hat{p}) + \bar{X}^A(\hat{q})) \leq \rho(\bar{X}^A(\hat{p})) + \rho(\bar{X}^A(\hat{q})) = \rho^A(\hat{p}) + \rho^A(\hat{q})$.
- SUB II From SUB II of C_T^A , anti-monotonicity, and subadditivity of ρ we have that $\rho^A(\hat{p} + \hat{q}) = \rho(\bar{X}^A(\hat{p} + \hat{q})) \leq \rho(\bar{X}^A(\hat{p}) + \bar{X}^A(\hat{q})) \leq \rho(\bar{X}^A(\hat{p})) + \rho(\bar{X}^A(\hat{q})) = \rho^A(\hat{p}) + \rho^A(\hat{q})$. But we know that $C_T^A(\hat{q}) = 0$, hence $\bar{X}^A(\hat{q}) = X(q)$ and the result follows.
- CTSUB Suppose $\hat{p} \in \mathcal{M}$, $\hat{q} \in \hat{\mathcal{P}}_{0,0} \subseteq \mathcal{M}$. Using translation anti-variance of ρ we have that

$$\begin{aligned} \rho^A(\hat{p} + \hat{q}) &= \rho(X(p) + X(q) - C_T^A(\hat{p} + \hat{q})) - \ell_0 - q_0 \\ &= \rho(X(p) - C_T^A(\hat{p} + \hat{q})) - \ell_0 - q_0, \end{aligned}$$

where we use the fact that $V_{0,0} = V_{0,T}$, hence $X(q) = 0$. Furthermore, we have that $\rho^A(\hat{p}) - q_0 = \rho(X(p) - C_T^A(\hat{p})) - \ell_0 - q_0$. Now the result follows directly from SUB II of C_T^A (Theorem 2.22) and anti-monotonicity of ρ .

SUP From SUP of C_T^A , anti-monotonicity, and additivity of ρ we get $\rho^A(\hat{p} + \hat{q}) = \rho(\bar{X}^A(\hat{p} + \hat{q})) \geq \rho(\bar{X}^A(\hat{p}) + \bar{X}^A(\hat{q})) = \rho(\bar{X}^A(\hat{p})) + \rho(\bar{X}^A(\hat{q})) = \rho^A(\hat{p}) + \rho^A(\hat{q})$.

AM The result follows directly from anti-monotonicity of ρ . \square

We see that convexity and positive super-homogeneity of risk measures is preserved under the liquidity adjustment on \mathcal{M} , while coherence is not, reflecting the common idea that size does matter. Convexity shows that even under liquidity risk the concept of risk diversification survives. Acerbi and Scandolo (2008) have a similar result and they rightfully emphasize that having convexity as a result rather than an axiom shows that generalizing coherence to convexity to deal with liquidity risk is not needed at the level of the underlying risk measure. It is worth mentioning that by Jensen's inequality convexity implies that the liquidity-adjusted risk of any convex combination of portfolio pairs is smaller than the convex combinations of the individual risks: for all $\hat{p}_1, \dots, \hat{p}_n \in \mathcal{M}$ and for all $\theta_i \in [0, 1]$ such that $\sum_{i=1}^n \theta_i = 1$, $\rho^A(\sum_{i=1}^n \theta_i \hat{p}_i) \leq \sum_{i=1}^n \theta_i \rho^A(\hat{p}_i)$. Note that VaR is not convex, hence using VaR as the underlying risk measure leads to a loss of convexity in positions.²⁸ Positive super-homogeneity confirms the common intuition (recall the earlier discussion) that the level of riskiness generally increases with increased position size when liquidity risk is present. Clearly, ρ^A is super-homogenous for any positive homogenous underlying risk measure ρ . Hence, VaR and any coherent risk measure, such as Expected Shortfall and more general spectral risk measures, inherit this property. A similar result is proven in Acerbi and Scandolo (2008) under Proposition 5.5.

Interestingly, general additivity in terms of portfolio pairs cannot be derived because of the opposing liquidity cost effects, discussed earlier (see Theorem 2.22). SUB I, SUB II, and CTSUP represent special cases where the diversification effect of liquidity sources dominates and works in the same direction as the subadditivity of ρ . CTSUP says that adding equity-funded cash initially has a positive effect that goes beyond benefits of adding an equity buffer. Because of the assumption that $V_{0,0} = V_{0,T} = 1$, we can credit this positive effect purely to a positive liquidity effect. The result follows from the observation that having extra cash in a scenario at T adds a frictionless source of liquidity for the generation of the liquidity call that spares the liquidation of less liquid assets and hence reduces the liquidity costs (cf., CTSUB of Theorem 2.22), but only on \mathcal{M} . A similar result shows up in Acerbi and Scandolo (2008) as “translationally subvariance”. The difference to our result is that we need to qualify the claim by saying that the funding source must be equity.

In contrast, the concentration effect of liquidity sources dominates if one considers

²⁸Under some practically relevant assumptions of the underlying probability distribution, VaR is subadditive in the tail region (Danielsson et al., 2005). If these conditions apply to the liquidity-adjusted situation as well, then L-VaR would of course be convex.

upscaling as reflected in PSUPH but also adding two portfolio pairs that have identical liquidity cost profiles almost surely, as stated in SUP . Notice, however, that SUP only holds if the underlying risk measure ρ is additive, which is very restrictive. If ρ would be subadditive, we would have that

$$\rho^A(\hat{p} + \hat{q}) \geq \rho(\bar{X}^A(\hat{p}) + \bar{X}^A(\hat{q})) \leq \rho^A(\hat{p}) + \rho^A(\hat{q}),$$

which leaves the the sign of overall effect of adding the two portfolio pairs ambiguous. This means that subadditivity of the underlying risk measures is not necessarily preserved under the liquidity adjustment, i.e., we can have that $\rho^A(\hat{p} + \hat{q}) \geq \rho^A(\hat{p}) + \rho^A(\hat{q})$, despite an underlying risk measure that is subadditive. In that case a bank wishing to take the risk $X^A(\hat{p}) + X^A(\hat{q})$ has the incentive to break up into two separately incorporated affiliates, one for the risk $X^A(\hat{p})$ and the other for the risk $X^A(\hat{q})$, because they would incur a smaller overall capital requirement of $\rho^A(\hat{p}) + \rho^A(\hat{q})$. It is standard to use the latter situation as a general argument for using subadditive risk measures. We think the argument is reasonable as long as opposing liquidity risk effects are not present.

The properties CONV , CTSUB , and SUB I above only hold on \mathcal{M} . This fact shows that Type 2 liquidity risk can disturb the workings of the diversification effect. This can lead to some seemingly unpleasant consequences. Consider we have some portfolio pair $\hat{p} \in \mathcal{P}$ and an equity-funded cash portfolio $\hat{q}_{0,0} \in \hat{\mathcal{P}}_{0,0}$ such that $(\hat{p}, \hat{q}) \notin \mathcal{M}$. For a risk measure that is sensitive to default by illiquidity states, it can very well happen that $\rho^A(\hat{p}) \leq \rho^A(\hat{p} + \hat{q}_{0,0})$. It appears odd at first sight that adding a riskless portfolio pair \hat{q} (no P&L risk, no funding risk, and no market liquidity risk) can actually lead to an increase in the riskiness. However, we believe this to be sensible as the risk measure balances the positive effects (cf., CTSUB) of adding the portfolio pair in the non-default states with the default states in which the added notional cash value is completely lost. We illustrate the link between diversification, subadditivity, and Type 2 liquidity risk in the following example.

Example 2.27 Subadditivity, convexity, and liquidity risk. Suppose we have two portfolio pairs $\hat{p}, \hat{q} \in \hat{\mathcal{P}}$ that have the same type of assets but different funding structure. The asset portfolio consists of cash, a liquid asset, and an illiquid asset, i.e., $N = 2$:

$$\hat{p} = ((p_0, p_1, p_2), \ell_{\hat{p}}) = ((0, 10, 10), \ell_{\hat{p}}) \quad \hat{q} = ((q_0, q_1, q_2), \ell_{\hat{q}}) = ((5, 10, 10), \ell_{\hat{q}}).$$

The initial unit MtM values are $V_{1,0} = 10$ and $V_{2,0} = 8$, hence $V_0(p) = 180$ and $V_0(q) = 185$. We assume market liquidity is modeled by exponential proceed functions with random friction parameters $\theta_{1,T}$ and $\theta_{2,T}$: $G_{i,T}(x_i) = \frac{V_{i,T}}{\theta_{i,T}}(1 - e^{-\theta_{i,T}x_i})$ for $i = 1, 2$. We assume we have linear liquidity call function A . Suppose three different scenarios $\Omega = \{\omega_1, \omega_2, \omega_3\}$ can occur at T . The scenarios and the corresponding probabilities are as follows:

$$\begin{aligned} \omega_1 &= (V_{1,T} = 11, V_{2,T} = 9, \theta_{1,T} = 0.02, \theta_{2,T} = 0.3, A(\hat{p}) = A(\hat{q}) = 0) \quad \mathbb{P}\{\omega_1\} = 0.95, \\ \omega_2 &= (V_{1,T} = 9.5, V_{2,T} = 8, \theta_{1,T} = 0.04, \theta_{2,T} = 0.5, A(\hat{p}) = 20, A(\hat{q}) = 40) \quad \mathbb{P}\{\omega_2\} = 0.04, \text{ and} \\ \omega_3 &= (V_{1,T} = 8.5, V_{2,T} = 7, \theta_{1,T} = 0.08, \theta_{2,T} = 0.4, A(\hat{p}) = 60, A(\hat{q}) = 70) \quad \mathbb{P}\{\omega_3\} = 0.01. \end{aligned}$$

The standard P&L X and the liquidity-adjusted P&L X^A in each scenario, as well as VaR

and ES at a 95% confidence interval, are given in the table below.

	$X(p)$	$X(q)$	$X(p+q)$	$X^A(\hat{p})$	$X^A(\hat{q})$	$X^A(\hat{p} + \hat{q})$
ω_1	20	20	40	20	20	40
ω_2	-5	-5	-10	-5.97	-8.14	-18.34
ω_3	-25	-25	-50	-52.26	-63.30	-365.00
VaR _{0.95}	5	5	10	5.97	8.14	18.34
ES _{0.95}	9	9	18	15.23	19.17	87.67

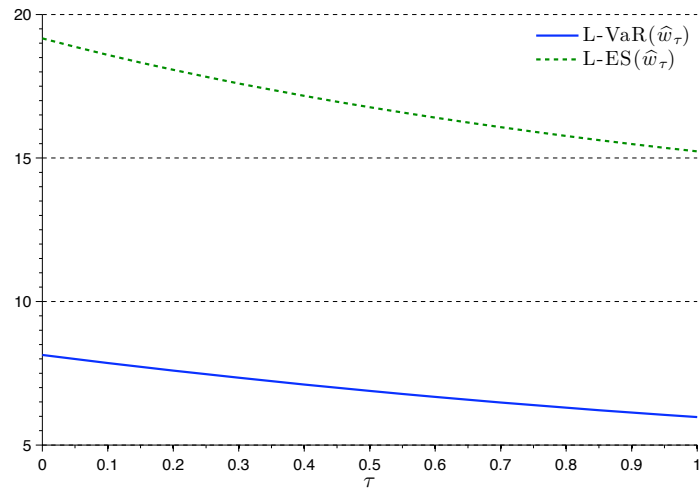
We can see that L-ES in this example is not subadditive in portfolio pairs, as

$$\text{L-ES}_{0.95}(\hat{p}) + \text{L-ES}_{0.95}(\hat{q}) = 34.40 < \text{L-ES}_{0.95}(\hat{p} + \hat{q}) = 87.67,$$

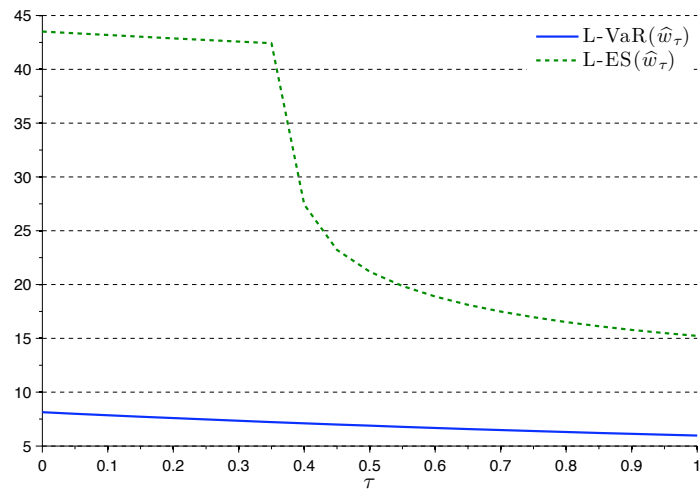
which can be explained by the negative concentration effect of liquidity sources of assets. What about convexity? Consider the convex combination of the two portfolio pairs: $\hat{w}_\tau = \tau\hat{p} + (1-\tau)\hat{q}$, $\tau \in [0, 1]$. Note that by linearity of A , we have that $A(\hat{w}_\tau) = \tau A(\hat{p}) + (1-\tau)A(\hat{q})$. On the left of Figure 2.3 we see that the risk surface of L-VaR_{0.95}(\hat{w}) and L-ES_{0.95}(\hat{w}), given the scenarios and probabilities as specified above, is convex for both risk measures. Now suppose we slightly change the third scenario by increasing the liquidity call of portfolio \hat{q} by 10, i.e., we have $\alpha_{\hat{q}} = 80$ instead of 70, and keep everything else the same. After the slight model change, we see on the right of Figure 2.3 that the risk surface of L-ES is not convex anymore. The lack of convexity can be explained by the fact that the model change brought the pair of portfolio pairs outside of \mathcal{M} , while before they were inside. The result is line with Theorem 2.26. Note that the risk surface of L-VaR is convex on the right-side as well. This fact can be explained by the model assumptions and the fact that at the chosen confidence level L-VaR is insensitive to the default by illiquidity states (see Equation 2.10). ■

Note that we can recover exactly the coherency axioms, if we assume that liquidity plays no role in any scenario. Acerbi and Scandolo (2008) refers to this case as the “limit of vanishing liquidity risk”. Two ways to achieve this limit in our setting is to have (1) zero liquidity calls almost surely, and/or (2) perfectly liquid assets almost surely. It is easy to see that in both cases the optimal liquidity costs are zero almost surely and we are back to the standard linear portfolio value setting $\rho^A(\hat{p}) = \rho(X(p))$. For readers interested in the proofs we refer to Proposition 5.4 in Acerbi and Scandolo (2008). The two cases nicely illustrate that in our formalism liquidity risk is only relevant in the intersection between market liquidity risk and funding liquidity risk.

In conclusion, we recommend banks to use a convex underlying risk measure for liquidity-adjusted risk calculations to ensure convexity in positions. In particular, we think coherent risk measures, such as Expected Shortfall, are a good choice because of their useful properties in the limit of vanishing liquidity risk. However, if VaR is used as the underlying risk measure for other, more practical reasons, then we can at least be ensured that possible increases in riskiness due to size effects are captured (see PSUBPH).



(a) $(\hat{p}, \hat{q}) \in \mathcal{M}$



(b) $(\hat{p}, \hat{q}) \notin \mathcal{M}$

Figure 2.3: Based on Example 2.27, the figure illustrates the risk surfaces for the convex combination of two portfolio pairs under slightly different models. On the left, we see that the risk surface is convex and we have that $\hat{p}, \hat{q} \in \mathcal{M}$, while on the right we have that $\hat{p}, \hat{q} \notin \mathcal{M}$ and hence the convexity of the risk surface is lost for L-ES (and coincidentally not for L-VaR). The results are in line with Theorem 2.26.

Remark 2.5.3. While choosing the “right” class of risk measures is important, in practice the problem of choosing a specific risk measure from that class remains. Different risk measures can lead to very different capital requirements, even within the same class. Unfortunately, we cannot hope to use mathematical arguments to single out a particular risk measure from a class. In practice financial supervisors and bank managers typically use VaR at a certain confidence level, despite the fact that it fails to be coherent, for a number of different reasons (see p. 19 in BIS (2009)). Some of these are more convincing than others. Undoubtedly, VaR gained popularity in practice due to its seemingly straightforward intuition as VaR can be phrased in words people are familiar with (cf., p. 153 in Acerbi (2004)). However, there is sometimes another line of reasoning presented that favors VaR over other risk measures. In practice, banks need to fall below a specific PD level to achieve a specific credit rating. With this in mind, it is sometimes maintained that VaR is a meaningful risk measure to determine EC but spectral risk measures like ES are not (Klaassen and van Eeghen, 2009, p. 9) because “it does not indicate how much capital an institution should possess in order to avoid insolvency with a chosen level of confidence.” However, if we look carefully, this point is actually not relevant. Clearly, if we assume that $\text{VaR}_\beta(p) = \ell_0$, then $\text{PD}_X = F_X(-\text{VaR}_\beta(p)) = \beta$, because $\text{VaR}_\beta(p) = F_X^{-1}(\beta)$. This might be straightforward, but using a coherent risk measure such as ES does not change anything conceptually, except that $\text{PD}_X = F_X(-\text{ES}_\beta(p)) = \beta'$, where $\beta' \leq \beta$, under the assumption that $\text{ES}_\beta(p) = \ell_0$ and ES exists. While we do not see any conceptual problems with using risk measures other than VaR, such as ES, for the determination of capital requirements, VaR has some practical advantages in terms of estimation and robustness. In practice, we have to face the problem that far into the tail data gets very sparse and using a risk measure that is very sensitive to the far tail of the loss distribution is problematic. While we could try to rely on extreme value theory the basic problem still remains. Of course this problem applies to VaR with a very high confidence level as well, but the problem is compounded by using ES or more generally spectral risk measures.

2.6 Allocating liquidity-adjusted EC and RAROC to business units

In this section we are concerned with the effects that liquidity risk has on the problem of decomposing a bank’s overall risk capital and performance measure into a sum of contributions of the bank’s business units (assets or sub-portfolios).²⁹ We show that the allocation problem with liquidity risk can be aligned with the existing literature on capital allocation in the liquidity-free setting, without major changes. However, this comes at the cost of losing the total allocation property.

²⁹Recall that EC allocation is not a physical or legal but a *virtual* allocation, just as EC itself is a virtual “required” capital to support the risk exposure.

2.6.1 The capital allocation problem

Given that a bank has specified a probability model for its P&L at time T and chosen a monetary risk measure as a basis for its capital determination problem, the overall EC without liquidity risk is given by $EC(p) = \rho(\sum_{i=0}^N p_i X_i)$. Now suppose that the bank is interested in decomposing the overall EC into a sum of contributions of their business units to identify risk concentrations and optimize its balance sheet. Without loss of generality, we assume for simplicity that the bank's i th business unit coincides with p_i . The EC required for the i th business unit, if it would not be part of the bank, hence often called the *standalone EC*, is with some abuse of notation given by

$$EC(p_i) := EC(0, \dots, p_i, \dots, 0) = \rho(p_i X_i).$$

If the bank uses a subadditive risk measure or if the risk measure is subadditive for the particular probability model, it is clear that the overall risk capital is less than the sum of the standalone ECs:

$$EC(p) \leq \sum_{i=0}^N EC(p_i).$$

Hence, there is a diversification effect at work and decomposing the bank's overall EC is more difficult than simply using the standalone ECs. How should we decompose/allocate the overall EC so that it adequately reflects the diversification effect that each business unit has. The problem of finding sound allocation rules is generally known under the header of the *capital allocation* problem.

In practice, (normalized) standalone risk contributions and (normalized) incremental risk contributions, the difference between the total risk with the business unit and the risk without it (with-without principle), are often used. In contrast, academics favor what is known as the *Euler allocation principle* (gradient allocation, marginal allocation), which takes the partial derivative of the overall portfolio risk with respect to the business unit scaled by the exposure as the risk contribution. Tasche (2000, 2008) shows that, under some differentiability assumptions, partial derivatives are the only risk contributions that provide the right information about the profitability of business units in some sense. Denault (2001) shows that using the partial derivatives is the only "fair" allocation scheme for a coherent risk measure with homogenous P&L, using the theory of nonatomic cooperative game theory. Finally, Kalkbrener (2005) show that the Euler allocation principle can be derived from a set of axioms.

2.6.2 Soundness and the Euler allocation principle

We want to apply the perspective of Tasche (2000, 2008) to our setting with liquidity risk. As a preparation, we briefly review the main idea. Tasche looks at the allocation problem from a portfolio optimization perspective. He argues that in case the allocated RAROC of a business unit is greater than the portfolio-wide RAROC, increasing the position of the business unit should improve, at least locally, the overall RAROC. We call this requirement *soundness* (Tasche uses the terms *suitability* and *RAROC compatibility*). Let us denote the EC allocated to business unit i by $EC(p_i | p)$ and let the RAROC

allocated to business unit i be given by

$$\text{RAROC}(p_i | p) := \frac{p_i E[X_i]}{\text{EC}(p_i | p)}.$$

We can express soundness formally in the liquidity risk-free setting as follows:

Definition 2.28 Soundness. We call $\text{EC}(p_i | p)$ *sound* if for all $i = 0, \dots, N$

$$\text{RAROC}(p_i | p) > \text{RAROC}(p) \implies \text{RAROC}(p_0, \dots, p_i + h, \dots, p_N) > \text{RAROC}(p)$$

for sufficiently small $h \neq 0$.

Tasche shows that the only way to achieve soundness in the standard setting is to equate $\text{EC}(p_i | p)$ to the partial derivative of the overall EC with respect to the i th business unit. Formally, the *Euler allocation* is given by

$$\text{EC}_{\text{Euler}}(p_i | p) := p_i \text{EC}'_i(p),$$

with

$$\text{EC}'_i(p) := \lim_{h \searrow 0} \frac{\text{EC}(p_0, \dots, p_i + h, \dots, p_N) - \text{EC}(p)}{h} = \frac{\partial \text{EC}}{\partial p_i}(p).$$

If the underlying risk measure ρ is positively homogenous of degree one, it follows from Euler's theorem of positive homogenous functions that

$$\text{EC}(p) = \sum_{i=0}^N \text{EC}_{\text{Euler}}(p_i | p). \quad (2.12)$$

The fact that the allocated risk contributions add up to the overall EC is often referred to as the *total allocation property*. As $\text{EC}_{\text{Euler}}(p_i | p)$ is the only sound allocation method, we know that

$$\text{RAROC}_{\text{Euler}}(p_i | p) := \frac{E[X_i]}{\text{EC}'_i(p_i)} \quad (2.13)$$

is the only sound definition for $\text{RAROC}(p_i | p)$.

We think that soundness has a clear and relevant economic interpretation. In particular, it meets the criteria suggested by Gruendl and Schmeiser (2007) that whatever allocation rule one may choose, the properties of this rule should be helpful in reaching the bank's goals. Here it is important to realize that there are two different perspectives in practice on how to use allocated EC and RAROC figures. In some banks allocated EC and RAROC figures are only used as a *decision support tool* for the group management, not the business units themselves. Under this view, business units managers use the unit's stand-alone RAROC for themselves, given by

$$\text{RAROC}(p_i) := \frac{E[X_i]}{\text{EC}(p_i)},$$

together with a prescribed hurdle rate. It is argued that this way business unit managers can focus purely on their core businesses and leave aspects of potential diversification benefits of their transactions to higher level management, as the latter is deemed to be more qualified to make such decisions. Consequently, $\text{RAROC}_{\text{Euler}}(p_i | p)$ must only be seen as a decision-support tool for the bank's group management. However, there are other banks that allow business units managers to directly use the allocated RAROC

figures. Tasche's concept of soundness nicely fits both perspectives as it is reasonable to assume that one goal of the business units and the bank's group management is to optimize its portfolio, i.e., ask the question which business line should be expanded or contracted. Note that soundness agrees with the first order condition of the optimization problem of maximizing the overall expected liquidity-adjusted P&L subject to a risk capital upper bound (Hallerbach (2004), Pflug and Roemisch (2007, p. 208)).

Remark 2.6.1. While soundness is a reasonable idea, it should be noted that the straightforward way to make capital budgeting decisions for lines of business, with or without liquidity risk, is to directly evaluate whether a different pricing policy or whether and to what extent expanding or contracting the business will lead to higher or lower risk-adjusted profitability of the bank as a whole, taking into account any operational constraints. Or even more directly the bank management could choose the solution to a suitable optimization problem. However, we assume here as in most academic literature that the question is not whether to allocate or not, but rather how it should be done, if one must.

2.6.3 Soundness, liquidity risk, and the Euler allocation principle

The question is now whether we can simply apply the idea of soundness to our formalism with liquidity risk. While it is possible, it is not entirely straightforward. We can easily spot some problems just looking at the definition of the portfolio L-RAROC:

$$\text{L-RAROC}(\hat{p}) := \frac{\text{L-EP}(\hat{p})}{\text{L-EC}(\hat{p})} = \frac{E[X^A(\hat{p})]}{\rho^A(\hat{p})} = \frac{\sum_{i=1}^N p_i E[X_i] - E[C_T^A(\hat{p})]}{\rho^A(\hat{p})},$$

where $\text{L-EP}(\hat{p})$ stands for the expected liquidity-adjusted P&L of the portfolio pair \hat{p} . It should be clear that in order to specify the liquidity-adjusted RAROC of a business unit we also need, in addition to an allocation rule for the denominator $\rho^A(\hat{p})$, a principle for the allocation of the numerator $\text{L-EP}(\hat{p})$. Without liquidity risk, $p_i E[X_i]$ is taken as the obvious contribution of a business unit to the expected P&L, but with liquidity risk we do not have a straightforward definition of the P&L of a business unit, because C_T^A is not linear in \hat{p} anymore. This brings us to another problem. Because C_T^A is positively super-homogenous (Theorem 2.22), we know that $E[C_T^A(\hat{p})]$ and $\rho^A(\hat{p})$ are not positively homogenous in portfolio pairs. Hence, Euler's theorem does not apply anymore and the total allocation property (cf., Equation 2.12) for the Euler allocation principle does not hold in the liquidity-adjusted case. Finally, the notion of a business units requires some explanation in our setting with liquidity risk. In the standard setting we considered p_i as representing the business unit and hence partial derivatives make sense. With liquidity risk, however, we take a *balance sheet viewpoint* and partial derivatives are less meaningful.

As a preparation for applying the idea of soundness to our formalism, we need some additional notation and definitions. We start with the definition of a business unit. Under the balance sheet perspective we define business units as asset and liability pairs. For a given $\hat{p} \in \hat{\mathcal{P}}$, we assume for simplicity that a business unit is represented by

the i th asset position p_i and a matching liability vector/funding mix ℓ^i , assigned by the group management to the business unit.³⁰

$$\widehat{p}_i := ((0, \dots, p_i, \dots, 0), \ell^i) \in \widehat{\mathcal{P}},$$

where we require that the sum of the $N + 1$ business units adds up to the whole portfolio pair, i.e., $\sum_{i=0}^N \widehat{p}_i = \widehat{p}$. In order to have well-defined directional derivatives we assume that asset and liability pairs are matched initially, i.e., $V_0(p) = L_0(\ell)$, where $L_0(\ell)$ is the notional value of the liability portfolio at time 0, given by $L_0(\ell) := \ell_0 + \sum_{j=1}^N \ell_j L_{0,j}$, with $L_{0,j} \geq 0$ for $j = 1, \dots, M$, representing the notional unit values of the liabilities. A special case is to set $\ell_j^i = \frac{V_{0,i}}{V_0(p)} \ell_j$ for $j = 0, \dots, M$ and $i = 0, \dots, N$, which amounts to assigning to each business unit the same funding mix. Given our definition of business units, we could define the liquidity-adjusted standalone EC and RAROC. However, we refrain from doing that here because we take the view that the type of liquidity risk we consider is a group's problem and hence managing it is not a core competency of a business unit. Furthermore, the issue involves some further difficulties. We refer the interested reader to Chapter 4 where we discuss the problem in more detail and offer a solution.

In contrast to the partial derivatives in the standard setting, we now consider for a given $\widehat{p} \in \widehat{\mathcal{P}}$ the directional derivatives in the directions of the i th business unit \widehat{p}_i/p_i :

$$\text{L-RAROC}'_i(\widehat{p}) := \lim_{h \searrow 0} \frac{\text{L-RAROC}(\widehat{p} + h \frac{\widehat{p}_i}{p_i}) - \text{L-RAROC}(\widehat{p})}{h}. \quad (2.14)$$

The directional derivatives of L-EP and L-EC at \widehat{p} are defined analogously and denoted by $\text{L-EP}'_i(\widehat{p})$ and $\text{L-EC}'_i(\widehat{p})$, respectively. Of course, the directional derivatives are the dot product of the corresponding *gradient* and the direction vector of the business unit \widehat{p}_i/p_i .

We denote the expected liquidity-adjusted P&L allocated to business unit \widehat{p}_i , given that it is part of portfolio pair \widehat{p} by $\text{L-EP}(\widehat{p}_i | \widehat{p})$ and the risk allocated to business unit \widehat{p}_i , given that it is part of portfolio pair \widehat{p} , by $\text{L-EC}(\widehat{p}_i | \widehat{p})$. The definition of the allocated L-RAROC follows:

$$\text{L-RAROC}(\widehat{p}_i | \widehat{p}) := \frac{\text{L-EP}(\widehat{p}_i | \widehat{p})}{\text{L-EC}(\widehat{p}_i | \widehat{p})}.$$

Applying the idea of soundness (see Definition 2.28) to the allocated L-RAROC, we call $\text{L-RAROC}(p_i | \widehat{p})$ *sound* if for all $i = 0, \dots, N$

$$\text{L-RAROC}(\widehat{p}_i | \widehat{p}) > \text{L-RAROC}(\widehat{p}) \implies \text{L-RAROC}(\widehat{p} + h \frac{\widehat{p}_i}{p_i}) > \text{L-RAROC}(\widehat{p}) \quad (2.15)$$

for a sufficiently small $h \neq 0$. Using L'Hôpital's rule, we can rewrite the right side of Equation 2.15 as

$$\frac{\text{L-EP}'_i(\widehat{p})}{\text{L-EC}'_i(\widehat{p})} > \frac{E[X^A(p)]}{\text{L-EC}(p)} \quad i = 1, \dots, N,$$

³⁰Again, the restrictions to a single p_i is not essential but simplifies some notation.

which gives us the sufficient condition for achieving L-RAROC:

$$\text{L-RAROC}(\hat{p}_i | \hat{p}) = \frac{\text{L-EP}(p_i | \hat{p})}{\text{L-EC}(p_i | \hat{p})} = \frac{\text{L-EP}'_i(\hat{p})}{\text{L-EC}'_i(\hat{p})}. \quad (2.16)$$

As a result, if we are interested in soundness, we need to comply to Equation 2.16. For example, if we would like to use the Euler allocation principle for the risk (denominator), we also need to use it for the numerator. While it appears that using the Euler allocation principle is also the way to go with liquidity risk, there is the problem that the allocated parts do not add up to the whole, hence failing the desirable *total allocation property*. In particular, the scaling constant $\eta_{\text{L-EP}}$, given by

$$\eta_{\text{L-EP}} \sum_{i=0}^N p_i \text{L-EP}'_i(\hat{p}) = \text{L-EP}(\hat{p}),$$

and $\eta_{\text{L-EC}}$, given by

$$\eta_{\text{L-EC}} \sum_{i=0}^N p_i \text{L-EC}'_i(\hat{p}) = \text{L-EC}(\hat{p}),$$

are generally not equal to one. The loss of the total allocation property is unfortunate but we could of course simply consider the normalized versions under a *proportional sharing rule* of the residual,³¹ given by

$$\overline{\text{L-EP}}(\hat{p}_i | \hat{p}) := p_i \text{L-EP}'_i(\hat{p}) \eta_{\text{L-EP}} \quad \text{and} \quad \overline{\text{L-EC}}(p_i | \hat{p}) := p_i \text{L-EC}'_i(\hat{p}) \eta_{\text{L-EC}}. \quad (2.17)$$

However, note that we cannot simply use the normalized versions of the numerator *and* the adjusted denominator for the allocated L-RAROC *at the same time* and still expect that soundness holds, because soundness then requires $\eta_{\text{L-EP}} = \eta_{\text{L-EC}}$, but in almost all cases this does not hold true.

We conclude that Tasche's notion of soundness can be applied to our formalism with liquidity risk without major problems. We only need to consider the appropriate directional derivatives, rather than use the partial derivatives. However, due to the inhomogeneity of L-EP and L-EC in business units we cannot achieve soundness and the total allocation property for both L-EP and L-EC *at the same time*.

Remark 2.6.2. The complications due to liquidity risk are caused by the inhomogeneity of X^A in \hat{p} . As it is standard in finance to assume that there is a linear relationship between exposure and values (see Footnote 10 on p. 8 and Equation 3.3 on p. 67), which is reasonable for risk types other than liquidity risk, it is not surprising that the majority of the capital allocation literature has not focused on the inhomogeneous case. However, Powers (2007) looked into the problem of an insurer interested in allocating its risk capital for losses that are inhomogeneous in exposures. The author argues for the use of *Aumann-Shapley (AS) values* due to its solid axiomatic foundation in nonatomic cooperative game theory and the fact that the total allocation property holds also for inhomogeneous losses. In fact Denault (2001) derives AS values as the only "fair" allocation rule for the capital allocation problem under coherent risk measures. However, in his case losses are homogenous in exposure and hence the AS values

³¹Note that it is also possible to share the residual equally among business units and not in proportion to marginal cost.

coincide with the Euler contributions. At first sight AS values seem to a good candidate for our formalism with liquidity risk. However, we believe that Tasche's notion of soundness is practically more relevant than the more abstract notion of "fairness" from which the AS values are derived, and unfortunately the allocated RAROC is not sound anymore if we use AS values for the numerator *and* the denominator.

2.7 Liquidity risk model calibration

Despite the simplicity of our framework, banks would still need to model proceed functions and liquidity call functions in order to apply our ideas. It is beyond the scope of this thesis to discuss the modeling task in detail, but we would like to point out some challenges.

The first challenge is to get hold of relevant liquidity data. In order to calibrate even simple proceed function models, for instance of the exponential form used in our examples, we would need time series of data of the form $(x_{i,t}, G_{i,t}(x_{i,t}))_{j=1}^n$ for $i = 1, \dots, N$ with $x_{i,t}$ being the quantity of asset i sold at time t and $G_{i,t}(x_{i,t})$ being the proceeds received at time t . However, as we are only interested in stress situations we need, in addition, to make sure that the data come from a period of market distress. While the recent Subprime crisis can provide some useful data points, data scarcity still remains a problem. An option might be to tie the proceeds functions of the bank to available market liquidity risk indices. For instance, the Bank of England's market liquidity index or Fitch's market liquidity indices on the CDS markets. Formally, we could assume co-monotonicity between the variations of the liquidity index and the proceed functions. In addition to the proceed functions, a bank needs to calibrate a liquidity call model. A potential problem is that each bank would ideally have to calibrate their models with bank-specific statistical data due to bank-specific factors such as market presence, crisis communication skills, market confidence etc. However, many banks did not experience idiosyncratic liquidity shocks in their past, so the relevant data set is small or worse empty. Of course, if the bank-specific data is not relevant, banks might extrapolate from the experience of other banks that experienced funding problems. This approach would be similar to the use of external data for the modeling of operational risk. In addition to these problems, we should also be concerned with the ever present issue of relying purely on historical data.

Even if the calibration to statistical liquidity data is possible, the next challenge is to specify the probabilistic dependence structure between liquidity risk, credit risk, market risk, and operational risk to complete the liquidity-adjusted EC model. Apart from ad-hoc approaches, we could imagine that it might prove useful to estimate a parametric multivariate copula between liquidity risk, market risk, and credit risk indices, as suggested by Brigo and Nordio (2010) in a different context and apply to our formalism.³² In Chapter 3 we propose a different approach that avoids the need to

³²While copulas have received a lot of attention in academic literature, it should be noted that banks often prefer to use simpler aggregation methods for the computation of EC due to practical reasons.

specify the probabilistic dependency between the standard risk types and liquidity risk separately.

In conclusion, we believe that at this point a solution based on the intelligent combination of data-driven EC models and expert-driven liquidity stress scenarios is attractive. In the future, useful statistical liquidity risk models might become more feasible due to the combined effort of market participants, regulators, and rating agencies to collect relevant financial data. As we propose the integration of liquidity risk into EC, we would require a formal incorporation of these liquidity stress scenarios into existing EC models, which is different from the current practice in the banking world to have a separate liquidity stress scenario analysis. A very simple approach to incorporate a particular notion of (liquidity) stress scenarios into the EC model is to consider a *finite mixture* between the standard EC model and a set of liquidity stress scenarios. The intuition is reasonable straightforward: we sample from the standard EC model with certain probability close to one to get ordinary and “moderately” pessimistic scenarios and a sample from a finite set of stress scenarios that include liquidity problems. Such an approach to incorporating stress scenarios into probability models is discussed in Berkowitz (2000) and Gibson (2001). While this approach to liquidity risk modeling is very simple, it has several advantages. Primarily it allows banks to think in terms of stress scenarios that involve not only the standard risk factors of the EC model but also liquidity risk. Furthermore, as banks are required to perform stress testing for the standard EC model by regulation anyway (BIS, 2005), banks might be able to “kill two birds with one stone”. Clearly, the resulting joint scenarios would be very pessimistic, and could be considered as “perfect storm” scenarios, but liquidity risk has exactly this character. We do not consider here how a bank comes up with the scenarios but we point into the direction of Breuer et al. (2009) for some ideas about the issue. A practical advantage of this modeling approach is that it is easy to sample realizations from the finite mixture using standard Monte Carlo techniques. Naturally, the specifications of the probabilities for the stress scenarios in the mixture is a problem in practice. One could build another (simple) model that allows banks to condition the stress probabilities on recent realizations of risk factors as is suggested in Berkowitz (2000) or leave it completely up to experts or regulators.

2.8 Simulation example

Suppose a bank’s asset portfolio consists of p_0 units of cash, p_1 units of a liquid asset, and p_2 units of an illiquid asset, i.e., $N = 2$.³³ The asset portfolio is funded by ℓ_0 units of equity, ℓ_1 units of retail on-demand deposits, ℓ_2 units of short-term wholesale funding, and ℓ_3 units of long-term bonds, i.e., $M = 3$:

$$\hat{p} = ((p_0, p_1, p_2), (\ell_0, \ell_1, \ell_2, \ell_3)) = ((10, 10, 10), (20, 30, 10, 4)),$$

³³We refer the reader to Chapter 3 for a less stylized illustration of our formalism.

with $V_{1,0} = 10$, $V_{2,0} = 8$, and $(L_0, L_1, L_2, L_3) = (1, 1, 10, 10)$. It is easily verified that the initial value of asset portfolio equals the value of the liabilities. For the calculation of the directional derivatives, we need to specify the matching portfolio pairs of the business units. For this example, we assign to each business unit the same funding mix such that the i th business unit is given by

$$\hat{p}_i = ((0, \dots, p_i, \dots, 0), (\tau \ell_0, \tau \ell_1, \tau \ell_2, \tau \ell_3)), \text{ with } \tau := \frac{p_i V_{0,i}}{V_0(p)}.$$

We assume that the marginal probability law of the two asset price is the log-normal distribution. In particular, we have that $V_{1,T} \sim \ln \mathcal{N}(2.35, 0.095)$ and $V_{2,T} \sim \ln \mathcal{N}(2.1, 0.067)$ so that $E[V_{1,T}] = 10.5$ and $E[V_{2,T}] = 8.2$. Furthermore, we assume for simplicity that both proceed functions have the exponential form and both have deterministic friction parameters $\theta_1 = 0.08$ and $\theta_2 = 0.1$:

$$G_{i,T}(x_i) = \frac{V_{i,T}}{\theta_i} (1 - e^{-\theta_i x_i}) \quad i = 1, 2.$$

We assume that the liquidity call function is linear in liabilities. More specifically, we assume that there is the possibility of a liquidity crisis with probability κ that amounts to a (constant) liquidity call given by the risk weighted *notional* liabilities. More formally, for a given constant risk weight vector $(w_1, w_2, w_3) \in [0, 1]^3$ and constant vector $(L_1, L_2, L_3) \in \mathbb{R}^3$ the liquidity call function is

$$A(\hat{p}) = \alpha = \left(\sum_{j=1}^3 w_j \ell_j L_j \right) C,$$

where $C \sim \text{Bernoulli}(\kappa)$ with κ being the probability of the liquidity crisis. We set κ to 0.01 and use the weighting vector $w = (0.5, 0.75, 0.25)$ so that $\alpha = 100C$. Note that equity does not carry any risk weight. Also note that because A is linear, we have that $A(\hat{p}_i) = \frac{p_i V_{0,i}}{V_0(p)} A(\hat{p}) = \frac{p_i V_{0,i}}{V_0(p)} 100C$. We complete the joint probability model by joining the marginal distributions with a Student's t copula with 1 degree of freedom and correlation matrix

$$\Sigma = \begin{bmatrix} 1 & 0.6 & -0.8 \\ 0.6 & 1 & -0.8 \\ -0.8 & -0.8 & 1 \end{bmatrix}.$$

We use VaR and ES to measure the bank's riskiness. Furthermore, we use the Euler allocation principle for the standard case and the liquidity-adjusted setting in line with soundness. We use $\text{VaR}_\beta(p_i | p)$ and $\text{L-VaR}_\beta(\hat{p}_i | \hat{p})$ to denote the risk allocated to the i th business unit if the underlying risk measure is VaR; the notation for ES is entirely similar. We approximate the partial and directional derivatives with the central difference, taking into account the rounding and truncation error (see the Appendix for the details), except for the ES contributions where use the formula given in Tasche (2008):

$$\text{ES}_\beta(p_i | p) \approx -E[p_i \hat{X}_i | \hat{X}(p) \leq -\text{VaR}_\beta(\hat{X}(p))],$$

where \hat{X}_i is the P&L of the i th business unit and $\hat{X}(p)$ the portfolio P&L under the "empirical measure". We apply VaR, ES, L-VaR (Equation 2.7), and L-ES (Equation 2.8) to

the sampled P&L in the standard way. In Figure 2.4 and Table 2.1 we collect the results of a simulation run of length 20,000. The first thing we notice is that the inclusion of liquidity risk increases the portfolio-wide riskiness by 8% for VaR and 58% for ES, as well as decreases portfolio RAROC by 3% using VaR and by 9% using ES. It should not be surprising that L-ES is considerably higher than ordinary ES as the optimal liquidity cost term is for the most part active far out in the tail in our example (see $\kappa = 0.01$). This aspect is also illustrated by the fact that the PD without liquidity risk ($PD_X = 0.023$) and with liquidity risk ($PD_{X^A} = 0.024$) is about the same. In terms of the risk contributions it is interesting to see that $L-ES(\hat{p}_0 | \hat{p})$ is negative. This result can be explained by the fact that cash is risk-free and, while it does not generate profit in our example, it decreases the optimal liquidity costs in all scenarios except those where the bank defaults due to illiquidity (see the discussion and example after Theorem 2.26). In this example the positive effects dominate the increased loss in the default states. The idea of soundness can nicely be illustrated by looking at $L-RAROC_{\text{Euler}}$ for the case of VaR. If we compare the overall liquidity-adjusted performance, given by $L-RAROC(\hat{p}) = 0.25$, with the allocated performance ratios, given by $L-RAROC_{\text{Euler}}(\hat{p}_1 | \hat{p}) = 0.27$ and $L-RAROC_{\text{Euler}}(\hat{p}_2 | \hat{p}) = 0.18$, we expect, in line with soundness (Equation 2.15), that increasing the first business unit and decreasing the second a bit would improve the overall performance. And indeed, we have $L-RAROC(\hat{p} + h \frac{\hat{p}_1}{\rho_1}) > 0$ and $L-RAROC(\hat{p} + h \frac{\hat{p}_2}{\rho_2}) < 0$ (not shown in Table 2.1). Finally, the example provides an illustration of the conflict between the total allocation property and soundness. As in (Equation 2.17), we define the scaling factors such that

$$\eta_{L-\text{VaR}} := \frac{L-\text{VaR}_{0.99}(\hat{p})}{\sum_{i=0}^2 L-\text{VaR}_{0.99}(\hat{p}_i | \hat{p})} \quad \eta_{L-\text{ES}} := \frac{L-\text{ES}_{0.99}(\hat{p})}{\sum_{i=0}^2 L-\text{ES}_{0.99}(\hat{p}_i | \hat{p})} \quad \eta_{L-\text{EP}} := \frac{L-\text{EP}(\hat{p})}{\sum_{i=0}^2 L-\text{EP}(\hat{p}_i | \hat{p})}.$$

As expected, we have that $\eta_{L-\text{VaR}} \neq \eta_{L-\text{ES}} \neq \eta_{L-\text{EP}}$, hence soundness and the total allocation property for both the numerator and the denominator at the same time cannot be achieved.

2.9 Discussion

2.9.1 Static versus dynamic framework

Practitioners might argue that our static approach neglects essential dynamic (timing) elements of liquidity risk. In particular, funding risk cannot be reduced to a single liquidity call number, as banks face a dynamic multi-period net cash flow balance that is a complex combination of cash in- and outflows streams arising from its asset and liability portfolio. In addition, recovery actions of banks facing serious liquidity problems are more complex and might involve actions of different durations. The feasibility of these measures are a function of the nature, severity, and duration of the liquidity shocks, in other words, it is a function of the state of the world and time. We agree that a dynamic analysis is conceptually more desirable than our simple static approach, but we would like to emphasize that the *costs* to produce it can be

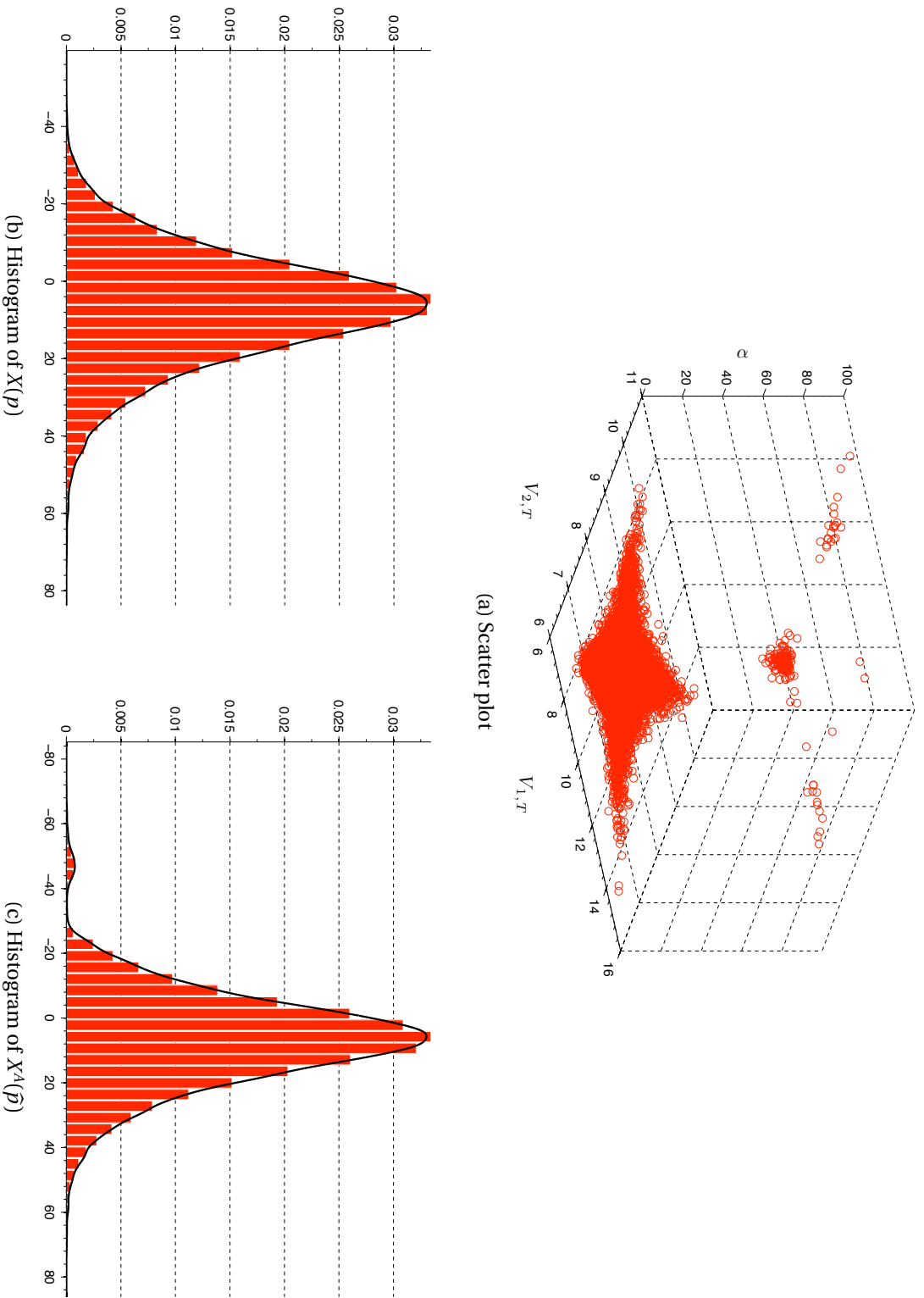


Figure 2.4: At the top, we see a scatter plot of the realizations of the asset prices at time T and the liquidity calls. At the bottom, we have the histograms of the frictionless and the liquidity-adjusted P&L with a kernel estimate of the density (see appendix in Chapter 3 for more information).

Risk measurement			
$VaR_{0.99}(p)$	25.07	$ES_{0.99}(p)$	29.74
$L-VaR_{0.99}(\hat{p})$	<u>27.13</u>	$L-ES_{0.99}(\hat{p})$	<u>46.87</u>
	+2.06		+17.13
PD_X	0.023	PD_{X^A}	0.024
Performance measurement			
$E[X(p)]$	6.97	$L-EP(\hat{p})$	6.77
VaR		ES	
$RAROC(p)$	0.28	$RAROC(p)$	0.23
$L-RAROC(\hat{p})$	<u>0.25</u>	$L-RAROC(\hat{p})$	<u>0.14</u>
	-0.03		-0.09
Risk allocation[†]			
$VaR_{0.99}(p_0 p)$	0	$L-VaR_{0.99}(\hat{p}_0 \hat{p})$	0
$VaR_{0.99}(p_1 p)$	15.65	$L-VaR_{0.99}(\hat{p}_1 \hat{p})$	17.11
$VaR_{0.99}(p_2 p)$	9.42	$L-VaR_{0.99}(\hat{p}_2 \hat{p})$	10.03
$ES_{0.99}(p_0 p)$	0	$L-ES_{0.99}(\hat{p}_0 \hat{p})$	-2.20
$ES_{0.99}(p_1 p)$	18.65	$L-ES_{0.99}(\hat{p}_1 \hat{p})$	42.69
$ES_{0.99}(p_2 p)$	11.09	$L-ES_{0.99}(\hat{p}_2 \hat{p})$	29.62
Performance allocation			
$p_0 E[X_0]$	0	$L-EP_{Euler}(\hat{p}_0 \hat{p})$	0.03
$p_1 E[X_1]$	4.96	$L-EP_{Euler}(\hat{p}_1 \hat{p})$	4.68
$p_2 E[X_2]$	2.01	$L-EP_{Euler}(\hat{p}_2 \hat{p})$	1.80
VaR			
$RAROC_{Euler}(p_1 p)$	0.32	$L-RAROC_{Euler}(\hat{p}_1 \hat{p})$	0.27
$RAROC_{Euler}(p_2 p)$	0.21	$L-RAROC_{Euler}(\hat{p}_2 \hat{p})$	0.18
ES			
$RAROC_{Euler}(p_1 p)$	0.27	$L-RAROC_{Euler}(\hat{p}_1 \hat{p})$	0.11
$RAROC_{Euler}(p_2 p)$	0.18	$L-RAROC_{Euler}(\hat{p}_2 \hat{p})$	0.06
Total allocation property and normalization			
$L-VaR_{0.99}(\hat{p})$	27.13	$L-ES_{0.99}(\hat{p})$	46.87
$\sum_{i=0}^2 L-VaR_{0.99}(\hat{p}_i \hat{p})$	<u>27.14</u>	$\sum_{i=0}^2 L-ES_{0.99}(\hat{p}_i \hat{p})$	<u>70.11</u>
	+0.01		+23.24
$L-EP(\hat{p})$	6.77	η_{L-VaR}	0.9996
$\sum_{i=0}^2 L-EP(\hat{p}_i \hat{p})$	<u>6.51</u>	η_{L-ES}	0.68
	-0.26	η_{L-EP}	1.04

[†] We refer the reader to the Appendix for details on how we approximate the derivatives numerically.

Table 2.1: Portfolio-wide EC and RAROC results and business unit allocation results.

considerable. Formalizing the above mentioned elements can very quickly become prohibitively complex. Mathematically speaking, modeling all multi-period (stochastic) cash in- and cash outflows of a bank's portfolio pair as well as specify multi-period recovery strategies, would require a multi-period stochastic optimization framework with a multi-variate stochastic process model that would easily exceed the existing EC model in complexity. Furthermore, ensuring the time-consistency of risk measures would be required, adding to the complexity (see, for instance, Artzner et al. (2007) and Roorda et al. (2005)). In this thesis we have chosen simplicity over completeness in the hope to illustrate the main ideas. Furthermore, we have argued several times in this chapter that it is possible to interpret the liquidity call in our framework as a bank's short-term cumulative net cash outflow under stress similar to the denominator in the LCR of Basel III.

2.9.2 Smooth cost term versus binary nature of liquidity risk

Our framework, and in particular, the optimal liquidity costs term, relies on the relevance of the implied optimization problem. We said that the liquidity call α should be interpreted as conditional on a crisis situation in which case the bank cannot access short-term unsecured funding and must rely on liquidating parts of its asset portfolio in illiquid secondary markets. Critics might say that we artificially introduce a "smooth" liquidity cost term where there is none in practice because under the assumed circumstances a bank could not hope to survive and would essentially be bankrupt before it even starts liquidating. The idea might better apply to special investment vehicles (SIVs). In our formalism we could express this line of reasoning by having either zero optimal liquidity costs or the bank is in default due to illiquidity: for all $\omega \in \Omega$, $X^A(\hat{p}, \omega) = X(p, \omega)$ or $X^A(\hat{p}, \omega) = -V_0(p, \omega)$. First of all, it should be noted that our formalism can deal with this special case and still produce meaningful L-EC and L-RAROC figures. However, it is true that under such an assumption the added value of our formalism would be rather minor. A possible response to the criticism is to stress that the assumption of no access to short-term unsecured funding should be interpreted as a short-term problem because it is clear that a bank cannot survive without access to unsecured funding over the medium-term. In addition, we can interpret α as being the liquidity call that is left after unsecured funding channels have been exhausted, which could be state-dependent. Another argument for the relevance of our framework is that the idea behind Basel III's LCR is essentially identical to our line of reasoning. However, if the sequence of events of a liquidity call and the subsequent fire selling in illiquid secondary asset markets really is not directly meaningful, we can still use the formalism as a consistent approach to reward and penalize banks for a reasonable notion of liquidity risk.

2.10 Conclusions

It has been the purpose of this paper to make EC and RAROC sensitive to liquidity risk in a consistent way, derive the basic properties of liquidity-adjusted risk measures, and address the problem of capital allocation under liquidity risk. We introduced the concept of optimal liquidity costs and liquidity cost profiles as a quantification of a bank's illiquidity at balance sheet level. This led to the concept of liquidity-adjusted risk measures defined on the vector space of asset and liability pairs under liquidity call functions.

In agreement with Key Observation 3 in the introduction, this formalism intrinsically combines market liquidity risk, reflected by the proceed functions, and funding liquidity risk, reflected by the liquidity call function. We showed that liquidity-adjusted risk measures possess reasonable properties under basic portfolio manipulations. In particular, we could show that convexity and positive super-homogeneity of risk measures is preserved under the liquidity adjustment, while coherence is not, reflecting the common idea that size does matter. Nevertheless, we argued that coherence remains a natural assumption at the level of *underlying* risk measures. Convexity shows that even under liquidity risk the concept of risk diversification survives. Positive super-homogeneity confirms the common intuition that the level of riskiness generally increases with increased position size when liquidity risk is present. We showed that liquidity cost profiles can be used to determine whether combining positions is beneficial or harmful. In particular, we have shown that combining positions with the same marginal liquidity costs generally leads to an increase of total liquidity costs. This effect works in opposite direction of the subadditivity of the underlying risk measure. Finally, we addressed the liquidity-adjustment of the well known Euler allocation principle. We could show that such an adjustment is possible without losing the soundness property that justifies the Euler principle. However, it is in general not possible to combine soundness with the total allocation property for both the numerator and the denominator in liquidity-adjusted RAROC.

Our results may have implications for financial regulations and banks. Liquidity-adjusted risk measures could be a useful addition to banking regulation and bank management, as they capture essential features of a bank's liquidity risk, can be combined with existing risk management systems, possess reasonable properties under portfolio manipulations, and lead to an intuitive risk ranking of banks. In fact, our framework may be seen as a more elaborate and rigorous version of the Liquidity Coverage Ratio of Basel III (BIS, 2010). Furthermore, combining our framework with the ideas of mark-to-funding in Brunnermeier et al. (2009) and "CoVaR" in Adrian and Brunnermeier (2009) may help regulators manage systemic risk, originating from bank's individual liquidity risk exposure. Internally, banks could use liquidity-adjusted Economic Capital and liquidity-adjusted RAROC, as well as the allocation schemes, to manage their risk-reward profile.

In our analysis the liquidity risk faced by banks has been inherently static: banks

faced a single liquidity call at some time horizon and had to optimally recover from it instantaneously. However, dynamic liquidity calls and recovery strategies as well as feedback effects would be more faithful to the complexity of liquidity risk in practice. While formalizing these features could quickly become prohibitively complex, developing a dynamic multi-period setting without succumbing to intractability is a topic for future research. The concept of random liquidity call functions was crucial in describing the interaction between funding and market liquidity risk. Realistic modeling of this concept is a critical, yet underdeveloped research topic. Finally, we think that addressing the link between the allocation of liquidity costs, as presented in our formalism, and funds transfer pricing frameworks used in practice by banks is an interesting topic for future research.

Appendix

A. Numerical approximation of the Euler allocations

For the computation of the risk contributions using the Euler allocation method, we need to find the partial derivatives for the standard case and the directional derivatives for the case with liquidity risk. Unfortunately, we need to rely on numerical approximations of the derivatives, because analytical solutions are not available for the liquidity risk case. While conceptually straightforward, the numerical approximation of the first derivative is not trivial in practice. Note that for the standard case we can rely on the results of Tasche (2008) as mentioned in Section 2.8.

Consider the derivative of some differentiable function f , given by

$$f'(x) := \lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h}.$$

The simplest way to approximate the derivative in x numerically is to either use the forward, backward, or central difference. The forward difference is given by

$$\hat{f}'(x) = \frac{f(x+h) - f(x)}{h},$$

the backward difference by

$$\hat{f}'(x) = \frac{f(x) - f(x-h)}{h},$$

and the central difference by

$$\hat{f}'(x) = \frac{f(x+h) - f(x-h)}{2h},$$

with $h > 0$ being sufficiently small. Higher order approximations naturally exist but are not feasible for our purposes as we deal with expensive Monte Carlo simulation. Using one of the above finite differences as a proxy, we are now facing the problem of choosing the small change, i.e., h . This choice involves the trade-off between what is known as the *rounding error* and the *truncation error*. A rounding error occurs whenever we represent an infinite real-number by a finite computer representation of

that number. It should be clear that not all real numbers can be represented *exactly* by a computer in *floating point arithmetic*. The approximate representation of some real numbers leads to the rounding error. Nowadays, with the advancement of the hardware, the rounding error is a problem only in some circumstances. One is that the rounding error could be compounded through a large number of computations and the second case involves what is known as the *catastrophic cancellation of digits*. While the floating-point difference of nearly equal numbers can be computed without error, such differencing magnifies the relative importance of the rounding errors already present in representation of those two numbers. Unfortunately, in numerical differentiation, we invite catastrophic cancellation since we approximate the derivative by the difference quotients shown above.

The truncation error stems from the higher terms in the Taylor series expansion, as we can easily show for the forward difference

$$f(x+h) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!} h^i$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi) \text{ with } \xi \in (x, x+h)$$

$$f'(x) - \frac{f(x+h) - f(x)}{h} = -\frac{h}{2} f''(\xi)$$

and the central difference

$$f'(x) - \frac{f(x+h) - f(x-h)}{2h} = -\frac{h^2}{6} f'''(\xi).$$

One can drive the optimal value for h that minimizes the rounding and the truncation error, but we do not discuss the details and refer the interested reader to Quarteroni et al. (2007) and Press et al. (2007). We use the central difference for all our numerical differentiation as well as the following h value:

$$h(x) = (\epsilon_m)^{\frac{1}{3}} x, \quad (2.18)$$

where x is the point at which the derivative is to be evaluated and ϵ_m is the *machine precision* of the floating point arithmetic system, which in our case is 2.2204×10^{-16} (output of the built-in Matlab function “eps”, using Matlab 2010a).

References

- Acerbi, C. (2004). Coherent representations of subjective risk-aversion. In Szegoe, G., editor, *Risk Measure for the 21st Century*, pages 147–207. Wiley Finance. 2.4.4, 2.5.3
- Acerbi, C. and Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, 8:681–692. 2.1, 2.2, 2.2, 8, 2.2, 19, 2.5.3, 2.5.4, 2.5.4
- Adrian, T. and Brunnermeier, M. K. (2009). Covar. <http://ssrn.com/abstract=1269446>. Working paper, Federal Reserve Bank of New York / Princeton University, May. 2.10
- Alfonsi, A., Fruth, A., and Schied, A. (2010). Optimal execution strategies in limit order books with general shape functions. *Quantitative Finance*, 10:143–157. 2.2
- Allen, F. and Carletti, E. (2008). The role of liquidity in financial crises. <http://ssrn.com/abstract=1268367>. Working paper, September. 2.2.2

- Almgren, R. (2003). Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance*, 10:1–18. 2.1, 2.2
- Almgren, R. and Chriss, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–29. 2.1, 2.2
- Anderson, W., Liese, M., and Weber, S. (2010). Liquidity risk measures. http://www.ims.nus.edu.sg/Programs/financialm09/files/weber_tut1b.pdf. Working paper. 2.1, 2.2, 2.2.3, 2.4.4, 20, 2.4.4, 2.5.1
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent risk measures. *Mathematical Finance*, 9:203–228. 2.1, 2.4.2, 2.5.3, 26
- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., and Ku, H. (2007). Coherent multiperiod risk adjusted values and bellman's principles. *Annals of Operations Research*, 152:5–22. 2.9.1
- Berkowitz, J. (2000). A coherent framework for stress-testing. *Journal of Risk*, 2:5–15. 2.7
- BIS (2005). International convergence of capital measurement and capital standards: A revised framework. Bank of International Settlements. 2.7
- BIS (2009). Range of practices and issues in economic capital frameworks. Bank of International Settlements. 2.4.4, 2.5.3
- BIS (2010). Basel 3: International framework for liquidity risk measurement, standards and monitoring. Bank of International Settlements. 2.10
- Bouchaud, J. (2009). Price impact. <http://arxiv.org/abs/0903.2428>. Entry for the upcoming Encyclopedia of Quantitative Finance. 2.2
- Breuer, T., Jandacka, M., Rheinberger, K., and Summer, M. (2009). How to find plausible, severe and useful stress scenarios. *International Journal of Central Banking*, 5:206–224. 2.7
- Brigo, D. and Nordio, C. (2010). Liquidity-adjusted market risk measures with stochastic holding period. Working paper, King's College London, October. 2.2.1, 2.7
- Brunnermeier, M., Crockett, A., Goodhart, C., Persaud, A., and Shin, H. S. (2009). The fundamental principles of financial regulation. *Geneva Reports on the World Economy*, 11. 2.4.4, 2.10
- Danielsson, J., Jorgensen, B. N., Sarma, M., and Vries, C. G. D. (2005). Sub-additivity re-examined: The case for value-at-risk. Working paper, Eurandom. 28
- Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, 4:1–34. 2.6.1, 2.6.2
- Duffie, D. and Ziegler, A. (2003). Liquidation risk. *Financial Analysts Journal*, 42. 2.3.1
- Engle, R. and Ferstenberg, R. (2007). Execution costs. *Journal of Portfolio Management*, 33:34–45. 2.1
- Foellmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, 6:429–447. 2.5.3
- Frittelli, M. and Gianin, E. (2002). Putting order in risk measures. *Journal of Banking and Finance*, 26:1473–1486. 2.5.3
- Gibson, M. S. (2001). Incorporating event risk into value-at-risk. Federal Reserve Board. 2.7
- Glosten, L. and Milgrom, P. (1985). Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics*, 14:71–100. 2.2
- Gruendl, H. and Schmeiser, H. (2007). Capital allocation for insurance companies: What good is it? *Journal of Risk and Insurance*, 74:301–317. 2.6.2
- Hallerbach, W. G. (2004). Capital allocation, portfolio enhancement and performance measurement: a unified approach. In Szegoe, G. P., editor, *Risk Measures for the 21st Century*, pages 435–450. John Wiley & Sons. 2.6.2
- Heath, D. and Ku, H. (2004). Pareto equilibria with coherent measures of risk. *Mathematical Finance*, 14:163–172. 2.5.3
- Jarrow, R. A. and Protter, P. (2005). Liquidity risk and risk measure computation. *Review of Futures Markets*, 11:27–39. 2.1, 2.2, 2.2.1, 2.5.3
- Jarrow, R. A. and Purnanandam, A. K. (2005). A generalized coherent risk measure from the firm's perspective. *Finance Research Letters*, 2:23–29. 2.4.1, 2.4.4
- Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance*, 15:425–437. 2.6.1
- Klaassen, P. and van Eeghen, I. (2009). *Economic Capital - How it works and what every manager should know*. Elsevier. 2.4.3, 2.4.4, 2.5.3
- Krokhmal, P. and Uryasev, S. (2007). A sample-path approach to optimal position liquidation. *Annals of Operations Research*, 152:193–225. 2.1

-
- Ku, H. (2006). Liquidity risk with coherent risk measures. *Applied Mathematical Finance*, 13:131–141. 2.1
- Kyle, A. S. (1985). Continuous auctions and insider trading. *Econometrica*, 53:1315–1336. 2.2
- Matz, L. and Neu, P. (2007). *Liquidity Risk Measurement and Management - A practitioner's guide to global best practices*. Wiley. 4, 21
- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press. 1, 2.2.1, 24
- Pflug, G. C. and Roemisch, W. (2007). *Modeling, Measuring, and Managing Risk*. World Scientific Publishing. 2.6.2
- Plantin, G., Shin, H. S., and Sapra, H. (2008). Marking to market: Panacea or pandora's box? *Journal of Accounting Research*, 46:435–460. 2.2
- Powers, M. R. (2007). Using aumann-shapley values to allocate insurance risk: The case of inhomogeneous losses. *North American Actuarial Journal*, 11:113–127. 2.6.2
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. (2007). *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press. 2.10
- Quarteroni, A., Sacco, R., and Saleri, F. (2007). *Numerical mathematics*. Springer. 2.10
- Rockafellar, R. T. and Wets, R. J.-B. (2004). *Variational Analysis*. Springer-Verlag. 2.5.2
- Roorda, B., Schumacher, J., and Engwerda, J. (2005). Coherent acceptability measures in multiperiod models. *Mathematical Finance*, 15:589–612. 2.9.1
- Schied, A. and Schoenborn, T. (2009). Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13:181–204. 2.1
- Schoenborn, T. (2008). *Trade execution in illiquid markets - optimal stochastic control and multi-agent equilibria*. PhD thesis, Technische Universität Berlin. 2.2
- Subramanian, A. and Jarrow, R. A. (2001). The liquidity discount. *Mathematical Finance*, 11:447–474. 2.1
- Tasche, D. (2000). Risk contributions and performance measurement. www-m4.ma.tum.de/pers/tasche/riskcon.pdf. Working paper, February. 2.6.1, 2.6.2
- Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the euler principle. In Resti, A., editor, *Pillar II in the New Basel Accord: The Challenge of Economic Capital*, pages 423–453. Risk Books. 2.6.1, 2.6.2, 2.8, 2.10

3

Illustration of the framework

In this chapter, we present an illustration of the liquidity risk formalism and the mathematical results derived in Chapter 2 in the context of a semi-realistic economic capital model, focusing on the impact of the balance sheet composition on liquidity risk. We show that even a simple but reasonable implementation of liquidity risk, based on a Bernoulli Mixture model, leads to non-trivial results. In particular, we show that liquidity risk can lead to a significant deterioration of capital requirements and risk-adjusted performance for banks with safe funding but illiquid assets, due to Type 2 liquidity risk, as well as banks with liquid assets but risky funding, due to Type 1 liquidity risk.

3.1 Introduction

In Chapter 2 we presented a mathematical framework to adjust EC and RAROC for a notion of liquidity risk. Here, our goal is to illustrate this framework by analyzing the impact of liquidity risk within the context of a realistic EC model of a bank, focusing on the sensitivity of liquidity-adjusted EC and RAROC to the asset and liability composition. We proceed as follows:

1. Specify typical balance sheet compositions for three types of banks: 1. retail banks, 2. universal banks, and 3, investment banks.
2. Introduce reasonable marginal risk models for market, credit, and operational risk of a bank.
3. Propose reasonable marginal models for market liquidity risk (proceed functions) and funding liquidity risk (liquidity call function).
4. Aggregate the marginal risks into an overall EC model via a suitable copula.

5. Compute L-EC and L-RAROC as described in Chapter 2 for the three bank types and investigate its similarities and differences.
6. Analyze the impact of the bank size on L-EC and L-RAROC.

The chapter is organized as follows: In Section 3.3 we describe the basic setting, including the main assumptions and the necessary formal preliminaries. In Section 3.2 we introduce the notion of a bank's portfolio composition and discuss the three bank types. In Section 3.4 we discuss how we model the market, credit, and operational risk of the banks. We then introduce some basic marginal probability models for the liquidity risk components (proceed functions and liquidity call function) in Section 3.5, as well as, illustrate the different liquidity risk characteristics of the three bank types by comparing the liquidity cost profiles as a function of the liquidity call. Afterwards in Section 3.6, we turn to the problem of aggregating the marginal risk models to the overall EC model with the help of a copula. Subsequently in Section 3.7, we present and discuss the results of the simulation. We conclude in Section 3.8 with a discussion about some of the implications we can drive from the illustration. Some technical details are left to the Appendix.

3.2 Portfolio weights, bank size, and bank types

For the illustration of the formalism we consider three generic types of banks: (1) a *retail bank* characterized by a large loan book and predominately retail deposit funding, (2) a *universal bank* characterized by a diversified mix of asset and liability types, and (3) an *investment bank* characterized by a large trading portfolio and predominately short-term whole-sale funding. We assume for simplicity that the most important difference between the three bank types are the portfolio weights. The portfolio weights of a bank is an important variable in our analysis and refers to *fraction of the total money invested* in the different asset and liability categories. In particular, on the asset side we consider three categories: (1) a liquidity portfolio (LP), (2) a trading book (TB),¹ and (3) a loan (banking) book (LB). On the liability side we consider we four different categories: (1) equity, (2) retail on-demand deposits, (3) short-term wholesale funding, and (4) long-term funding. We abstract from “other assets” and off-balance sheet items. We assign the following portfolio weights with the three bank types:

Retail bank				Universal bank			
Liquidity pf	15%	Equity	5%	Liquidity pf	15%	Equity	5%
Loan book	70%	Deposits	60%	Loan book	45%	Deposits	45%
Trading book	15%	Short-term	20%	Trading book	40%	Short-term	35%
		Long-term	15%			Long-term	15%
	100%		100%		100%		100%

¹By the trading book we mean all positions held intentionally for short-term profit-taking. It is the bank's proprietary trading position and not the positions arising from client servicing and market making.

Investment bank			
Liquidity pf	10%	Equity	8%
Loan book	15%	Deposits	5%
Trading book	75%	Short-term	67%
		Long-term	20%
	100%		100%

While it is reasonable to assume that the three bank types differ in overall size expressed as total assets, we first consider for simplicity that all three bank types have the size, to keep the analysis more streamlined. Note that the overall bank size is an important variable, when liquidity risk is present, as we have shown in Chapter 2. We will later on analyze the impact of bank size in the context of this illustration. We assume that all banks have total assets worth € 500bn. Given the overall bank size, we have the following balance sheets in terms of money invested and expressed in the notations introduced below:

Retail bank				Universal bank			
$V_0^{\text{lp}}(p^{\text{lp}})$	€ 75bn	ℓ_0	€ 25bn	$V_0^{\text{lp}}(p^{\text{lp}})$	€ 75bn	ℓ_0	€ 25bn
$V_0^{\text{lb}}(p^{\text{lb}})$	€ 350bn	$L_0^{\text{dep}}(\ell^{\text{dep}})$	€ 300bn	$V_0^{\text{lb}}(p^{\text{lb}})$	€ 225bn	$L_0^{\text{dep}}(\ell^{\text{dep}})$	€ 225bn
$V_0^{\text{tb}}(p^{\text{tb}})$	€ 75bn	$L_0^{\text{st}}(\ell^{\text{st}})$	€ 100bn	$V_0^{\text{tb}}(p^{\text{tb}})$	€ 200bn	$L_0^{\text{st}}(\ell^{\text{st}})$	€ 175bn
		$L_0^{\text{lt}}(\ell^{\text{lt}})$	€ 75bn			$L_0^{\text{lt}}(\ell^{\text{lt}})$	€ 75bn
	€ 500bn		€ 500bn		€ 500bn		€ 500bn

Investment bank			
$V_0^{\text{lp}}(p^{\text{lp}})$	€ 50bn	ℓ_0	€ 40bn
$V_0^{\text{lb}}(p^{\text{lb}})$	€ 75bn	$L_0^{\text{dep}}(\ell^{\text{dep}})$	€ 25bn
$V_0^{\text{tb}}(p^{\text{tb}})$	€ 375bn	$L_0^{\text{st}}(\ell^{\text{st}})$	€ 335bn
		$L_0^{\text{lt}}(\ell^{\text{lt}})$	€ 100bn
	€ 500bn		€ 500bn

More formally, we assume as in Chapter 2 that a bank's position is represented by a asset/liability pair (portfolio pair) $\hat{p} = (p, \ell) \in \hat{\mathcal{P}}$. We assume that a bank's asset portfolio $p \in \mathcal{P}$, expressed as units of contracts, consists of (1) a liquidity portfolio (cash/cash equivalents and very liquid assets), (2) a trading book, and (3) a loan (banking) book:

$$p = (p^{\text{lp}}, p^{\text{lb}}, p^{\text{tb}}) \in \mathcal{P} = \mathbb{R}_+^{N+1}$$

with $p^{\text{lp}} = (p_0, p_1^{\text{lp}}, \dots, p_{n_1}^{\text{lp}}) \in \mathbb{R}_+^{n_1+1}$, $p^{\text{lb}} = (p_1^{\text{lb}}, \dots, p_{n_2}^{\text{lb}}) \in \mathbb{R}_+^{n_2}$, and $p^{\text{tb}} = (p_1^{\text{tb}}, \dots, p_{n_3}^{\text{tb}}) \in \mathbb{R}_+^{n_3}$, so that $n_1 + n_2 + n_3 = N$. Each asset position has a corresponding unit price/value, e.g., the initial value of one unit of the i th asset in the loan book is denoted by $V_{i,0}^{\text{lb}} \geq 0$. We refer to the initial value of (or money invested in) the liquidity portfolio, the trading book, and the loan book by $V_0^{\text{lp}}(p^{\text{lp}})$, $V_0^{\text{tb}}(p^{\text{tb}})$, and $V_0^{\text{lb}}(p^{\text{lb}})$, respectively. We have that $V_0^{\text{lp}}(p^{\text{lp}}) := p_0 + \sum_{i=1}^{n_1} p_i^{\text{lp}} V_i^{\text{lp}}$. The others are defined analogously. The initial value of the whole asset portfolio is then given by

$$V_0(p) := V_0^{\text{lp}}(p^{\text{lp}}) + V_0^{\text{tb}}(p^{\text{tb}}) + V_0^{\text{lb}}(p^{\text{lb}}).$$

The asset portfolio weights are given by

$$\phi_{lp} := \frac{V_0^{lp}(p^{lp})}{V_0(p)} \quad \phi_{lb} := \frac{V_0^{lb}(p^{lb})}{V_0(p)} \quad \phi_{tb} := \frac{V_0^{tb}(p^{tb})}{V_0(p)}.$$

We assume that the bank's liability portfolio consists of (1) equity, (2) retail on-demand deposits, (3) short-term wholesale funding, and (4) long-term funding:

$$\ell = (\ell_0, \ell^{\text{dep}}, \ell^{\text{st}}, \ell^{\text{lt}}) \in \mathbb{R}_+^{M+1},$$

with $\ell_0 \in \mathbb{R}_+$, $\ell^{\text{dep}} = (\ell_1^{\text{dep}}, \dots, \ell_{m_1}^{\text{dep}}) \in \mathbb{R}_+^{m_1}$, $\ell^{\text{st}} = (\ell_1^{\text{st}}, \dots, \ell_{m_2}^{\text{st}}) \in \mathbb{R}_+^{m_2}$, and $\ell^{\text{lt}} = (\ell_1^{\text{lt}}, \dots, \ell_{m_3}^{\text{lt}}) \in \mathbb{R}_+^{m_3}$ so that $m_1 + m_2 + m_3 = M$. Similar to the asset side, we assume that each liability position has an initial exposure per unit, which corresponds to the notional value per unit of the liability, e.g, the initial notional of the j th deposit is denoted by $L_j^{\text{dep}} \geq 0$. We refer to the initial notional value of the deposits, the short-term funding, and the long-term funding by $L_0^{\text{dep}}(\ell^{\text{dep}})$, $L_0^{\text{st}}(\ell^{\text{st}})$, and $L_0^{\text{lt}}(\ell^{\text{lt}})$, respectively. We have that $L_0^{\text{dep}}(\ell^{\text{dep}}) := \sum_{j=1}^{m_1} \ell_j^{\text{dep}} L_j^{\text{dep}}$. The others are defined analogously. The initial notional value of the whole liability portfolio is then given by

$$L_0(\ell) := \ell_0 + L_0^{\text{dep}}(\ell^{\text{dep}}) + L_0^{\text{st}}(\ell^{\text{st}}) + L_0^{\text{lt}}(\ell^{\text{lt}}).$$

We assume that the initial value of asset portfolio equals the value of the liabilities: $V_0(p) = L_0(\ell)$. The portfolio weights of the liabilities are given by

$$\phi_{\text{equity}} := \frac{\ell_0}{L_0(\ell)} \quad \phi_{\text{dep}} := \frac{L_0^{\text{dep}}(\ell^{\text{dep}})}{L_0(\ell)} \quad \phi_{\text{st}} := \frac{L_0^{\text{st}}(\ell^{\text{st}})}{L_0(\ell)} \quad \phi_{\text{lt}} := \frac{L_0^{\text{lt}}(\ell^{\text{lt}})}{L_0(\ell)}.$$

As a result the *portfolio weights* of a bank are given by

$$\phi = ((\phi_{lp}, \phi_{lb}, \phi_{tb}), (\phi_{\text{equity}}, \phi_{\text{dep}}, \phi_{\text{st}}, \phi_{\text{lt}})) \in [0, 1]^7,$$

and the portfolio weights of the three bank types discussed above are then given by

$$\begin{aligned} \phi^{\text{ret}} &:= ((15\%, 70\%, 15\%), (5\%, 60\%, 20\%, 15\%)) \\ \phi^{\text{umi}} &:= ((15\%, 45\%, 40\%), (5\%, 45\%, 35\%, 15\%)) \\ \phi^{\text{inv}} &:= ((10\%, 15\%, 75\%), (8\%, 5\%, 67\%, 20\%)). \end{aligned}$$

We will explain the position sizes of the various categories in the following sections because they are related to the way we model the uncertainty.

3.3 Setting

3.3.1 Static risk measurement setup

For the computation of the riskiness we need to specify the profit and loss (P&L) distribution for each bank. We adopt the same basic setting as in Chapter 2 with only slight changes. We consider two moments in time: today denoted by $t = 0$ and some risk management horizon denoted by $t = T$, which we take to be 1 year. We assume that a bank's position today leads to overall P&L at time T , denoted by X . This models the future P&L as a random variable X at T , where $X(\omega)$ represents the profit

($X(\omega) > 0$) or loss ($X(\omega) < 0$) at time T , if the scenario ω realizes. More formally, fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathcal{H} := L^1(\Omega, \mathcal{F}, \mathbb{P})$ the space of all integrable random variables on (Ω, \mathcal{F}) . We interpret \mathcal{H} as the space of all conceivable future P&L of a bank over the time horizon T . Risk measures are defined as mappings from this space to the real line: $\rho : \mathcal{H} \rightarrow \mathbb{R}$. More specifically, we focus in this chapter on the monetary risk measures VaR and Expected Shortfall: for all $X \in \mathcal{H}$

$$\text{VaR}_\beta(X) := \inf\{c \in \mathbb{R} \mid \mathbb{P}\{-X \leq c\} \geq \beta\}, \quad (3.1)$$

and

$$\text{ES}_\beta(X) := \frac{1}{1-\beta} \int_\beta^1 \text{VaR}_u(X) du. \quad (3.2)$$

As explained in Chapter 2 we assume that there is a *linear* relationship between the exposure and the individual P&L of the different books, except for the liquidity cost term and operational losses. The linearity assumption makes sense in the typical specifications of market risk (trading book) and credit risk (banking book) but is not entirely convincing for operational risk because it is not a “position risk”.² While it is reasonable to assume that operational losses increase with position size, i.e., larger banks have a chance of higher losses, saying that losses scale linearly with the business unit size is not convincing. For simplicity we assume in our analysis that operational losses are independent of the position. For the nonlinear character of the liquidity cost term we refer the reader to Chapter 2. Furthermore, we do as if we confront the bank’s current position ($\hat{p} \in \hat{\mathcal{P}}$) with scenarios modeled as occurring at T . That way the position is the same today as it is at T . The assumption is simplistic but standard. We define the overall P&L of a bank without liquidity risk as the sum of the P&L of the different books:

$$X(\hat{p}) := X^{\text{lp}}(p^{\text{lp}}) + X^{\text{lb}}(p^{\text{lb}}) + X^{\text{tb}}(p^{\text{tb}}) - L^{\text{Op}}, \quad (3.3)$$

where L^{Op} stands for the cumulative operational losses over the period $(0, T]$. Furthermore, we have (the others are defined analogously) that $X^{\text{lp}}(p^{\text{lp}}) := \sum_{i=1}^{n_1} p_i^{\text{lp}} X_i^{\text{lp}}$, with $X_i^{\text{lp}} \in \mathcal{H}$ for all i . In addition, we assume that $L^{\text{Op}} \in \mathcal{H}$ which means that $X(\hat{p}) \in \mathcal{H}$. We do not model ALM risk consistent with the setting in Chapter 2.

Remark 3.3.1. In this chapter we use a confidence level of 99.99% for the computation of VaR and ES. However, in practice some banks use a confidence level of 99.9999% instead for the computation of economic capital.

3.3.2 Liquidity risk adjustment

Following Chapter 2, we assume that at time T liquidity problems in the form of liquidity calls might realize and the bank is forced to liquidate part of its asset portfolio. This extension leads naturally to random optimal liquidity costs at time T , which we denote by $C_T^A(\hat{p})$. The idea is to subtract, per scenario $\omega \in \Omega$, the liquidity costs incurred from

²However, as mentioned earlier under the Basic Indicator Approach and the Standardized Approach of Basel II, capital charges scale linearly with a bank’s overall and business line gross income, respectively.

generating the liquidity call $A(\hat{p}) = \alpha$ from the standard P&L at T :³

$$X^A(\hat{p}) := X(\hat{p}) - C_T^A(\hat{p}). \quad (3.4)$$

Notice that in a scenario in which the bank cannot meet the liquidity call and hence defaults (Type 2 liquidity risk), we assume that it incurs a loss equal to its full initial asset portfolio value: $\hat{p} \notin \mathcal{L}^A(\omega) \implies X^A(\omega) = -V_0(p)$. Essentially this means that we view $-V_0(p)$ as an upper bound for the capital losses that can occur over $(0, T)$. Formally, the liquidity adjustment requires the specification of the liquidity call function $A : \hat{p} \mapsto \alpha$. In addition, we also need N random asset proceed functions, each taking values in \mathcal{G} , i.e., the measurable functions $G_{i,T} : \Omega \rightarrow \mathcal{G}$ for all i . We now turn to the modeling of the P&L of the asset categories and afterwards we discuss ways to model liquidity risk components.

3.4 Modeling market, credit, and operational risk

3.4.1 Liquidity portfolio

We assume that the liquidity portfolio consists of cash and a government bond (default-free zero-coupon bond) with maturity $T = 10$.⁴ We refer to the cash position by p_0^{lp} and the bond position by p_1^{lp} so that $p^{\text{lp}} = (p_0^{\text{lp}}, p_1^{\text{lp}})$ for all three bank types. For simplicity, we assume that cash does not earn any interest.⁵ While we assume that government bonds are free of risk, as is standard, the price of the bonds is not constant due to volatile interest rates. For modeling the price changes of the bonds, we assume that the interest rate follows a mean-reverting stochastic process as introduced in Vasicek (1977), which is commonly called a *Vasicek process*.⁶ More formally, we denote the price of a bond at time t with maturity T_i by $B(t, T_i)$. The interest rate follows a Vasicek process:

$$dr_t = a(b - r_t)dt + \sigma_r dW_t,$$

where b is the equilibrium value that the rate reverts to in the long-run with speed a , σ_r is the instantaneous volatility, and dW_t is the increment of a Wiener process. Given that the interest rate follows a Vasicek process, the price of a bond with a *unit notional* becomes (see González et al. (2003) for the details)

$$B(t, T_i) = e^{m_t - n_t r_t}$$

with

$$n_t := \frac{1 - e^{-a(T-t)}}{a}$$

and

$$m_t := \frac{(n_t - T + t)(a^2 b - \frac{\sigma_r^2}{2})}{a^2} - \frac{\sigma_r^2 n_t^2}{4a}.$$

³For the definition of the optimal liquidity costs we refer the reader to Chapter 2.

⁴Note that the maturity is beyond the risk measurement horizon of $T = 1$ year.

⁵We could easily change that, so that the value of the cash account at time T would be $p_0^{\text{lp}} e^{r_r T}$.

⁶Not to be confused with the Vasicek credit risk model.

b	a	r_0	T	σ_r
0.05	0.5	0.04	10	0.13

Table 3.1: Parameters used for the Vasicek interest rate process in the simulation.

We assume that the notional of the bond is € 1,000,000 and hence we have that $V_{1,0}^{\text{lp}} = 10^6 B(0, 10) = 804,400$. One can show that the log of the bond price is normally distributed:

$$\ln B_i(t, T_i) \sim \mathcal{N} \left(m_t - n_t(b + (r_0 - b)e^{-at}), \frac{\sigma_r^2 n_t^2}{2a}(1 - e^{-2at}) \right).$$

The price of the bond at time $T = 1$ is then given by $V_{1,T}^{\text{lp}} = 10^6 B(T, T_1) = 10^6 B(1, 10)$. We assume for simplicity that the MtM changes of the liquidity portfolio goes through the bank's P&L account and hence increase/decrease the bank's capital. Note that this assumption may not entirely be in line with current accounting convention where often only realized P&L are registered, e.g., upon the sale of the bonds. However, as discussed in the previous chapter taking an economic perspective and using the fair value approach rather than taking the accounting perspective is reasonable in the EC context. Thus, the P&L of the liquidity portfolio is given by

$$X^{\text{lp}}(p^{\text{lp}}) = p_1^{\text{lp}}(V_{1,T}^{\text{lp}} - V_{1,0}^{\text{lp}}) = 10^6 p_1^{\text{lp}}(B(1, 10) - B(0, 10)).$$

It is convenient to express the P&L as a fraction of the initial capital, i.e.,

$$\bar{X}^{\text{lp}}(p^{\text{lp}}) := \frac{V_{1,T}^{\text{lp}} - V_{1,0}^{\text{lp}}}{\ell_0}.$$

We assume that 10% of the money invested according to the portfolio weight ϕ_{lp} in the liquidity portfolio is put into cash (p_0^{lp}), while the rest is invested in the government bond position (p_1^{lp}) so that

$$p_0^{\text{lp}} = \frac{0.1\phi_{\text{lp}}V_0(p)}{V_{0,0}^{\text{lp}}} = \phi_{\text{lp}}50 \text{ bln}$$

and

$$p_1^{\text{lp}} = \frac{0.9\phi_{\text{lp}}V_0(p)}{V_{1,0}^{\text{lp}}} = \frac{450,000}{B(0, 10)}\phi_{\text{lp}}.$$

The latter changes with the portfolio weights, given that we keep the model fixed. The descriptive statistics and the histogram of the marginal P&L distribution of the liquidity portfolio for the three bank types are presented in Figure 3.1 and Figure 3.2.

3.4.2 Loan book and credit risk

We assume that the loan book of each bank consists of loans belonging to one of five categories of credit worthiness. Generally, these categories can be interpreted as internal or externally assigned credit ratings. We use a mixture model to describe the credit risk for each of the categories. In a mixture model the default risk of the obligor

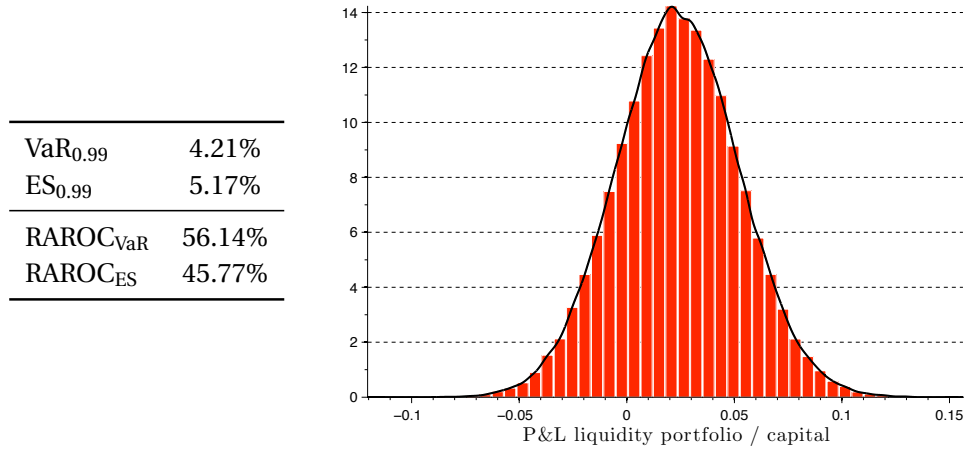


Figure 3.1: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the liquidity portfolio P&L divided by the initial capital for the retail and universal bank.

is assumed to depend on a set of common, usually observable, risk factors, which are taken to be random variables. Given a realization of the risk factors, defaults of individual obligors are taken as independent. As a result, the dependence between defaults is purely driven by the dependence of the common risk factors. McNeil et al. (2005) show that the standard credit risk model used for regulatory capital is a special case of the mixture model class discussed here.

We assume each loan $j \in \{1, \dots, n_2\}$ has a position size of p_j^{lb} and notional size of $V_{j,0}^{\text{lb}}$, so that $V_0(p^{\text{lb}}) = \sum_{j=1}^{n_2} p_j^{\text{lb}} V_{j,0}^{\text{lb}}$. The credit risk for the bank of loan j is that the obligor defaults between $(0, T]$. Suppose that the Bernoulli random variable Y_j stands for the binary event of default happening ("1") or not ("0") for loan j , i.e., $Y_j : \Omega \rightarrow \{0, 1\}$. The recovery amount per unit of the loan j is given by $V_{j,0}^{\text{lb}} \phi_j$, $\phi_j \in (0, 1]$. We assume for simplicity that ϕ_j is deterministic and the same for all j , hence we can write $\phi_j = \phi$. The random credit loss of loan j is then given by

$$L_j^{\text{Cr}} := (1 - \phi) V_{j,0}^{\text{lb}} Y_j.$$

Now suppose that all n_2 loans can uniquely be assigned to k different homogenous groups of loans representing, e.g., rating classes according to internal or external classification, with $n_2 > k$ and $r = 1, \dots, k$ indexing the categories. Let us write p_r^{lb} for the number of loans in category r . The loss per category r is then given by

$$L_r^{\text{Cr}}(p_r^{\text{lb}}) := \sum_{u=1}^{p_r^{\text{lb}}} L_u^{\text{Cr}}.$$

The total losses over the period $(0, T]$ of the loan book is then the sum of losses of all categories:

$$L^{\text{Cr}}(p^{\text{lb}}) := \sum_{r=1}^k L_r^{\text{Cr}}(p_r^{\text{lb}}). \quad (3.5)$$

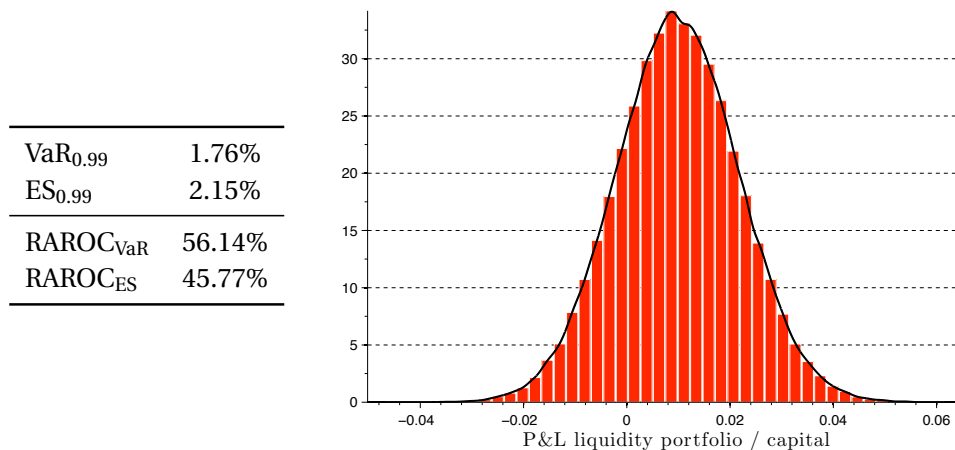


Figure 3.2: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the liquidity portfolio P&L divided by the initial capital for the investment bank. Note that the only difference between the investment and the other two banks is the amount of initial capital.

In order to proceed, we need to specify the joint distribution of the indicator variables Y . In our opinion, the mixture model class popular in credit risk modeling is a good candidate. In particular, we consider Bernoulli mixture models:

Definition 3.1 Bernoulli mixture model (McNeil et al., 2005). Given some random vector $Z = (Z_1, \dots, Z_n)'$ taking values in \mathbb{R}^n , the random vector $Y = (Y_1, \dots, Y_m)'$ follows a Bernoulli mixture model with factor vector Z , if there are functions $q_i : \mathbb{R}^n \rightarrow [0, 1]$ for $1 \leq i \leq m$, such that conditional on Z the indicator Y is a vector of independent Bernoulli random variables with $\mathbb{P}(Y_i = 1 | Z = z) = q_i(z)$. For $y = (y_1, \dots, y_m)' \in \{0, 1\}^m$ we have

$$\mathbb{P}(Y = y | Z = z) = \prod_{j=1}^m q_j(z)^{y_j} (1 - q_j(z))^{1-y_j}.$$

Let us write M_r for the (random) number of defaults (default counts) in the category r over the time interval $(0, T]$. We can write the Bernoulli mixture model as a binomial mixture model so that the default counts M_1, \dots, M_r are independent and the conditional distribution is binomial, i.e., $M_r | Z = z \sim \mathcal{B}(p_r^{\text{lb}}, q_r(z))$, where p_r^{lb} is the number of loans in category r . One way to specify the functions $q_r(z)$ is to use *generalized linear mixed models* (GLMMs). Suppose we have a vector of deterministic covariates $w_r \in \mathbb{R}^e$, consisting of characteristics of the obligor such as industry membership, geographic region, a regression vector $v_r \in \mathbb{R}^e$, an intercept parameter $\mu_r \in \mathbb{R}$, and a scaling vector $\sigma_r \geq 0$, then a model for the conditional default probability is

$$q_r(z) = h(\mu_r + v_r' w_r + \sigma_r' Z), \quad (3.6)$$

where $h : \mathbb{R} \rightarrow (0, 1)$ is a strictly increasing link function. Typical examples for link functions are the *standard normal distribution function* (probit link) and the *logistic distribution function* (logit link).

For simplicity, we use the model and estimated parameters in McNeil et al. (2005)

on page 381-382. More specifically, we consider a single-factor Bernoulli mixture model with a random effect representing the “state of the economy”, where the conditional default probability of category r is given by

$$q_r(z) = \Phi(\mu_r + Z), \quad (3.7)$$

with Φ being the standard normal distribution function and $Z \sim \mathcal{N}(0, \sigma^2)$. McNeil et al. (2005) estimate the model from Standard&Poor default count data for the five rating ($k = 5$) categories A to CCC. We reproduce the estimation parameter of McNeil et al. (2005) in Table 3.2. The maximum likelihood estimates of the standard deviation (scaling parameter) σ is 0.24. As p_r^{lb} stands for the number of loans in category r we have, e.g., that $M_1 | Z = z \sim \mathcal{B}(p_1^{\text{lb}}, \Phi(-3.43 + Z))$. For simplicity we assume that all loans have the same notional value V_0^{lb} of € 500,000 and the recovery rate ϕ is 40% so that the overall losses of the loan book are given by

$$L^{\text{Cr}}(p^{\text{lb}}) = \sum_{r=1}^k (1 - \phi) V_0^{\text{lb}} M_r = 300,000 \sum_{r=1}^k M_r.$$

As we are interested in the P&L of the bank and not only the losses, we assume that the bank earns the expected losses plus 0.4% on the money invested. As a result the P&L of the loan book are given by⁷

$$X(p^{\text{lb}}) = E[L^{\text{Cr}}(p^{\text{lb}})] + 0.004 V_0^{\text{lb}} - L^{\text{Cr}}(p^{\text{lb}})$$

We again express the P&L as a fraction of the initial capital, i.e.,

$$\bar{X}^{\text{lb}}(p^{\text{lb}}) := \frac{X^{\text{lb}}(p^{\text{lb}})}{\ell_0}.$$

We assume that the money invested according to the portfolio weight ϕ_{lb} is equally distributed over the five loan categories:

$$\frac{p_1^{\text{lb}} V_0^{\text{lb}}}{\sum_{r=1}^5 p_r^{\text{lb}} V_{0,r}^{\text{lb}}} = \dots = \frac{p_5^{\text{lb}} V_0^{\text{lb}}}{\sum_{r=1}^5 p_r^{\text{lb}} V_{0,r}^{\text{lb}}} = \frac{1}{5}$$

so that

$$p_r^{\text{lb}} = \frac{1}{5} \frac{\sum_{r=1}^5 p_r^{\text{lb}} V_{0,r}^{\text{lb}}}{V_0^{\text{lb}}} = \frac{1}{5} \frac{\phi_{\text{lb}} V_0(p)}{V_0^{\text{lb}}} = \frac{1}{5} \frac{\phi_{\text{lb}} 500 \text{ bln}}{500,000} = 200,000 \phi_{\text{lb}} \text{ for } r = 1, \dots, 5.$$

The descriptive statistics and the histogram of the marginal P&L distribution of the loan book for the three bank types are presented in Figure 3.3, Figure 3.4, and Figure 3.5.

Remark 3.4.1. While the simulation of the credit risk losses is rather straightforward, we need to sample from the binomial distribution with a very large number of trials (number of loans per category). Technically, this is no problem but doing this directly is computationally very expensive. Fortunately, we can use a normal approximation for our case, based on de Moivre-Laplace theorem. For the details we refer the reader to the Appendix.

⁷Alternatively, a bottom-up approach would require the specification of the bank's stipulated gain per loan or loan category if it is repaid regularly.

Parameter	A	BBB	BB	B	CCC
μ_r	-3.43	-2.92	-2.40	-1.69	-0.84
π_r	0.0004	0.0023	0.0097	0.0503	0.2078

Table 3.2: Reproduction of Table 8.8 on page 382 in McNeil et al. (2005): Maximum likelihood estimates for a one-factor Bernoulli mixture model fitted to historical default count data from Standard&Poor and the implied unconditional default probabilities π_r .

VaR _{0.99}	51.35%
ES _{0.99}	62.68%
RAROC _{VaR}	10.91%
RAROC _{ES}	8.93%

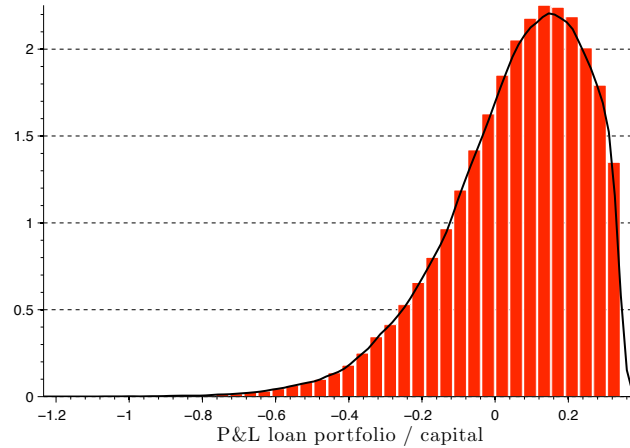


Figure 3.3: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the loan portfolio P&L divided by the initial capital of the retail bank.

3.4.3 Trading book and market risk

The P&L of the trading book over $(0, T)$ is given by the fair value change of the book:

$$X^{\text{tb}}(p^{\text{tb}}) := V_T^{\text{tb}}(p^{\text{tb}}) - V_0^{\text{tb}}(p^{\text{tb}}),$$

so that the P&L as a fraction of the initial capital is given by

$$\bar{X}^{\text{tb}}(p^{\text{tb}}) := \frac{X^{\text{tb}}(p^{\text{tb}})}{\ell_0}.$$

For simplicity, we use a reduced-form approach for modeling the trading book. More specifically, we assume that $n_3 = 1$ so that we can write $V_T^{\text{tb}}(p^{\text{tb}}) = p^{\text{tb}} V_T^{\text{tb}}$. We model the fair value of the trading book at time T as a log-normal distribution: $V_T \sim \ln \mathcal{N}(\mu_{\text{tb}}, \sigma_{\text{tb}})$ with $\mu_{\text{tb}}, \sigma_{\text{tb}} \geq 0$ so that the expected return on capital, i.e., $E[\bar{X}^{\text{tb}}(p^{\text{tb}})]$ is 3%. As before, the position size p^{tb} for each bank depends on the portfolio weights. For simplicity, we assume that the initial value for the trading position is 1, i.e., $V_0^{\text{tb}} = 1$, so that the money invested according to the portfolio weights is given by

$$p^{\text{tb}} = \phi_{\text{tb}} V_0(p) = \phi_{\text{tb}} 500 \text{ bln.}$$

The descriptive statistics and the histogram of the marginal P&L distribution of the trading book for the three bank types are presented in Figure 3.6, Figure 3.7, and Figure 3.8.

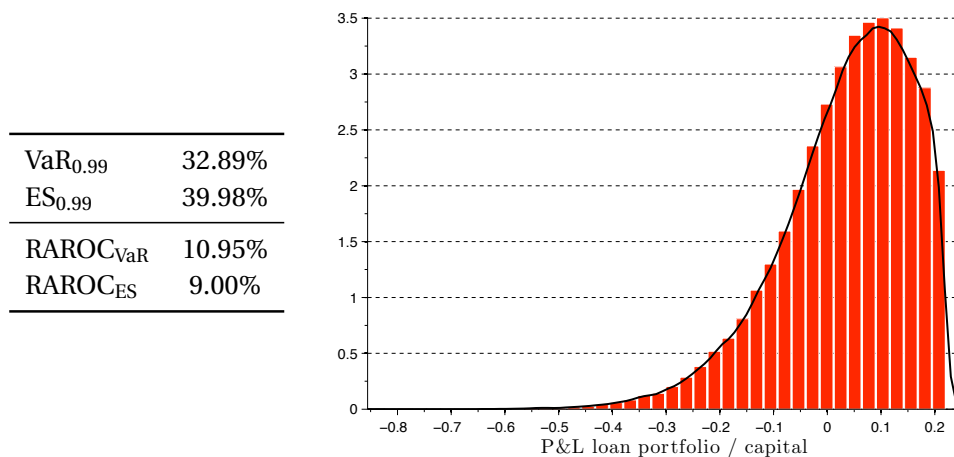


Figure 3.4: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the loan portfolio P&L divided by the initial capital of the universal bank.

3.4.4 Operational risk

We assume that all three types of banks are exposed to operational risk. It is reasonable to assume that there are differences in exposure between the three bank types, despite our assumption that all three banks have the same overall size. In particular, retail banks have less exposure to traditional sources of operational risk such as processing, custody-related activities, and rogue trading incidents than universal and investment banks. For simplicity we assume that all banks face the same operational risk as it is difficult to explicitly connect the position sizes with operational losses in our context (cf., earlier discussion about this issue). We do not consider the advanced modeling approach under Basel II but instead follow the simpler approach of Rosenberg and Schuermann (2006). We model the size of the operational losses of the bank over the time horizon $(0, T)$ by a Pareto distribution and the occurrence frequency by a binomial distribution.

In particular, consider $x := \ln(L^{\text{Op}})$, where L^{Op} is the operational loss corrected for a reporting threshold and assume that $x \sim b^{-1}e^{-x/b}$ with $b \geq 0$. Note that if $\ln(L^{\text{Op}})$ is exponentially distributed, then L^{Op} has a Pareto distribution. For the frequency distribution we assume that the binomial distribution has success probability $s \geq 0$ with 365 trials, assuming $T = 1$ year. We use for all three banks the parameters $b = 0.55$ and $s = 65/365$. Again, we express the operational losses as a fraction of the initial capital:

$$\bar{X}^{\text{Op}}(\hat{p}) := \frac{L^{\text{Op}}(\hat{p})}{\ell_0}.$$

The descriptive statistics and the histogram of the marginal distribution of the operational losses for the three bank types is presented in Figure 3.9.

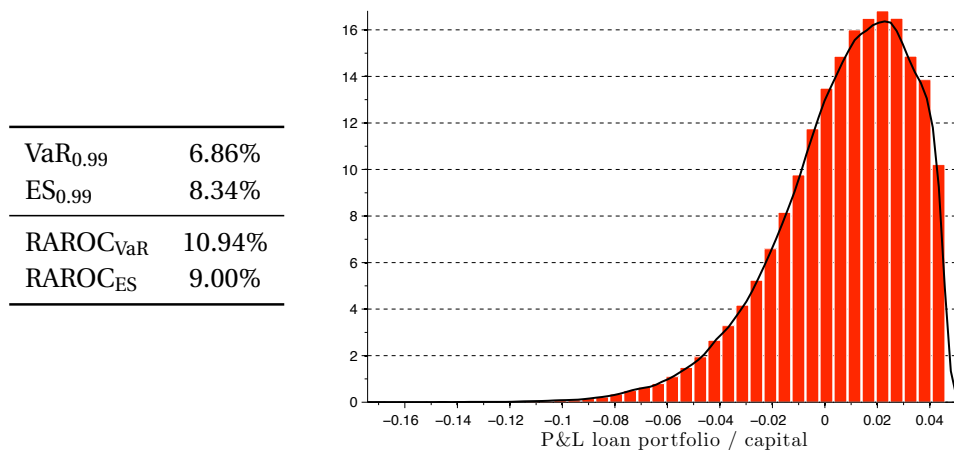


Figure 3.5: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the loan portfolio P&L divided by the initial capital of the investment bank.

3.5 Modeling liquidity risk

While it is beyond the scope of this thesis to discuss the liquidity risk modeling task in great detail, we aim to introduce some simple, yet reasonable probability models for the liquidity call function and the proceed functions. A typical way in finance to think of the risk of an entity is to conceptualize it as a function of idiosyncratic and systemic factors. This line of reasoning is also useful for liquidity risk as it is common practice in liquidity stress scenario analyses to think in terms of idiosyncratic and systemic stress scenarios (Matz and Neu, 2007). We would like to express this dichotomy in our liquidity modeling. More specifically, we let the probability of funding problems (liquidity calls) depend on changes in the BIS ratio of the bank. In scenarios with large decreases in the ratio, the probability of positive liquidity calls is greater. Due to the fact that the BIS ratio depends on the P&L distribution of the bank, we internalize the funding liquidity risk and avoid having to specify the probabilistic dependency separately.

3.5.1 Liquidity call function

We apply the idea behind the (Bernoulli) mixture model introduced earlier for credit risk modeling to the problem of modeling the liquidity call function. We assume that given the realization of the BIS ratio change, i.e., the idiosyncratic risk factor, the probability of a set of liquidity call scenarios is independent. As a result, the dependence between liquidity calls is purely driven by the dependence of the common risk factors, i.e., the BIS ratio change.⁸ The intuition is that solvency problems, represented by the

⁸A natural extension to our setting would be adding an additional random systemic risk factor. This extra term would allow us to model funding effects that do not originate from a bank's idiosyncratic solvency problems but from systemic effects such as contagious reputation damages.

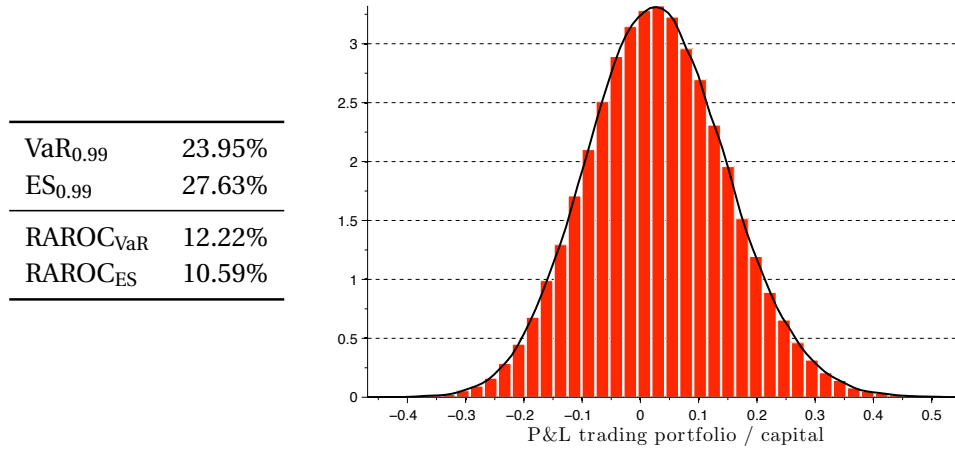


Figure 3.6: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the trading portfolio P&L divided by the initial capital of the retail bank.

downward changes of the BIS ratio, precede funding problems. Consequently, we assume that the conditional probability of liquidity calls is higher in scenarios in which the *downward change* of the BIS ratios is large.⁹ In practice, the BIS ratio is defined as

$$\text{BIS ratio} := \frac{\text{Tier I+II capital}}{\text{RWA}},$$

where RWA are the risk-weighted assets. We do not distinguish between the types of capital and we use

$$\text{BIS}_0 := \frac{\ell_0}{0.75V_0^{\text{lb}}(p^{\text{lb}}) + V_0^{\text{tb}}(p^{\text{tb}})} \quad \text{and} \quad \text{BIS}_T := \frac{\ell_0 + X(\hat{p})}{0.75V_0^{\text{lb}}(p^{\text{lb}}) + V_0^{\text{tb}}(p^{\text{tb}})}, \quad (3.8)$$

as a proxy for the initial and the random BIS ratio at time T , respectively. The numerator is the bank's capital per scenario and the denominator is the bank's initial RWA, where we assume that the liquidity portfolio has zero weight, the trading book has a weight of one, and the loan book has a weight of 0.75.¹⁰ However, we are primarily interested in the *downward percentage change* in the ratio over a year, i.e.,¹¹

$$\Delta_{\text{BIS}} := \frac{\text{BIS}_0 - \text{BIS}_T}{\text{BIS}_0} = -\frac{X(\hat{p})}{\ell_0},$$

and

$$\Pi := \max(0, \Delta_{\text{BIS}}) = \max\left(0, -\frac{X(p)}{\ell_0}\right), \quad (3.9)$$

⁹In a dynamic setting, it would be more reasonable to work with the rate of change of the BIS ratio over the time interval, rather than simply the difference between the start and the end values, as a slow and steady decline sends a different signal to fund providers than an abrupt decline.

¹⁰In practice, one assigns different weights to different types of loans, e.g., mortgages receive lower weights than corporate loans. We do not make these distinctions.

¹¹Alternatively, we could let the type of losses (credit loss, market loss, op-loss) have different impacts on the conditional liquidity call probability.

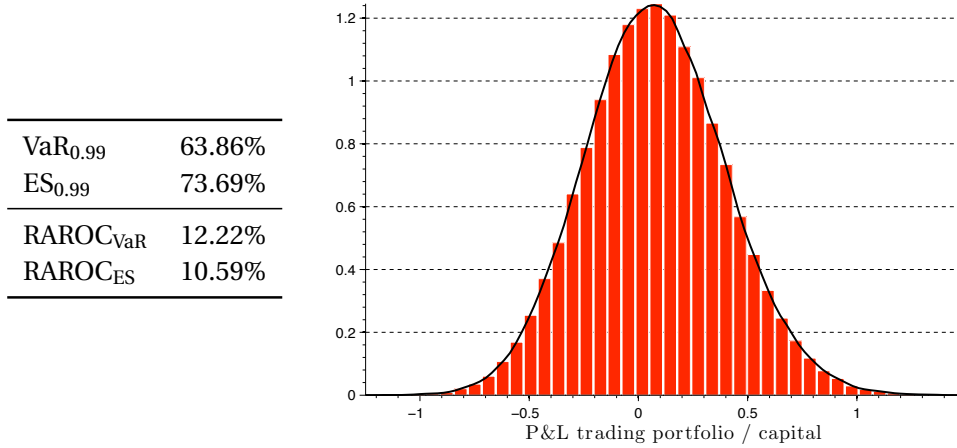


Figure 3.7: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the trading portfolio P&L divided by the initial capital of the universal bank.

so that the RWA specification drops out.¹² Now suppose there are $S > 1$ different liquidity call scenarios, indexed by $s = 1, \dots, S$. Each scenario s occurs with a probability pr_s^Π conditional on a realization of the BIS ratio change $\Pi = \pi$, so that

$$\text{pr}_s^\Pi = f_s(\pi) \text{ with } \Pi \sim F,$$

where F is the distribution function of Π and link function $f_s : [0, 1] \rightarrow [0, 1]$ for $s = 1, \dots, S$. Notice that the liquidity call scenarios S only depend on the EC scenarios via Π . We assume that we have unit positions in liabilities, i.e., $\ell^{\text{dep}} = \ell^{\text{st}} = \ell^{\text{lt}} = 1$, so that the notional values for each liability category is simply

$$L_0^d = \phi_d L_0(\ell) = \phi_d 500 \text{ bln for } d \in I := \{\text{dep, st, lt}\}.$$

We assume that the liquidity call in scenario s is given by $\sum_{d \in I} L^d w_d^s$, with $w_d^s \in (0, 1]$ per liability class $d \in I$. Note that capital, i.e., ℓ_0 , does not carry any risk weight. We assume for simplicity that w_d^s are deterministic for all d and s but differ among liability classes and scenarios. The risk weights should reflect the riskiness of the funding category, taking into account contractual optionalities, payment schedules, investor diversification, general “stickiness”, etc. The liquidity call sizes can then be interpreted as the (liquidity) risk-weighted liabilities, akin to the notion of RWA used in Basel I and II, but now per scenario. The conditional probability distribution function of α is then given by

$$(\alpha | \Pi = \pi) = \begin{cases} 0, & \text{with probability } \text{pr}_0^\Pi = 1 - \sum_{s=1}^S \text{pr}_s^\Pi \\ \sum_{d \in I} w_d^1 L_0^d, & \text{with probability } \text{pr}_1^\Pi = f_1(\pi), \\ \vdots & \\ \sum_{d \in I} w_d^S L_0^d, & \text{with probability } \text{pr}_S^\Pi = f_S(\pi), \end{cases} \quad (3.10)$$

¹²Note that this happens only because we use the initial RWA in the denominator of BIS_T .

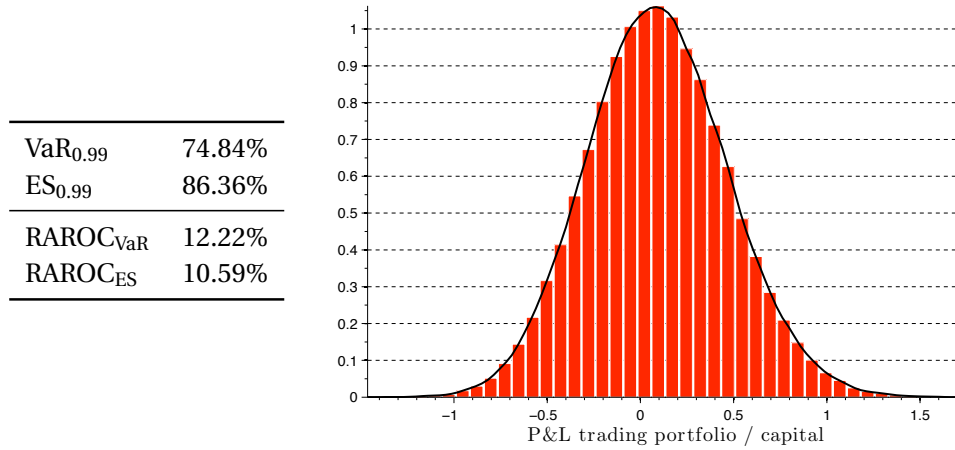


Figure 3.8: Descriptive statistics (left) and histogram with a kernel density estimate overlay (see Appendix) of the distribution of the trading portfolio P&L divided by the initial capital of the investment bank.

Note that the liquidity call function has the properties required as described in Definition 2.20 and is *linear* in liabilities:

$$A(\widehat{p}) = \alpha = \sum_{s=1}^S \left(\sum_{d \in I} w_d^s L_0^d \right) \mathbf{1}_{\{s\}},$$

where $\mathbf{1}_{\{s\}}$ is the indicator function for the s 's liquidity call scenario. For the link function we assume that f_1, \dots, f_S have the same functional form but may have different parameters. In particular, we assume that

$$\text{pr}_s^\Pi = f_s(\pi) := \text{Beta}(\pi, a_s, b_s) \text{ for } s = 1, \dots, S, \quad (3.11)$$

where $\text{Beta} : [0, 1] \rightarrow [0, 1]$ is the Beta cumulative distribution function with fixed shape parameters $a_s > 0$ and $b_s > 0$ for scenario s . Note that we cannot directly use $\text{Beta}(\pi, a_s, b_s)$ as the conditional probabilities because we need to normalize them first.

For our analysis we consider two liquidity call scenarios, i.e., $S = 2$: a moderate and an extreme stress scenario. We set the risk weights corresponding to these scenarios to

$$\text{moderate : } (w_{\text{dep}}, w_{\text{st}}, w_{\text{lt}}) = (0.10, 0.64, 0.08)$$

$$\text{extreme : } (w_{\text{dep}}, w_{\text{st}}, w_{\text{lt}}) = (0.26, 0.64, 0.24).$$

We have chosen the risk weights based on some intuitive order constraints among bank types and liability classes. The weights lead to in the following liquidity call sizes:

	Retail	Universal	Investment
moderate $\frac{\alpha}{V_0(p)}$	20.00%	28.10%	44.98%
extreme $\frac{\alpha}{V_0(p)}$	32.00%	37.70%	48.98%

Note that the risk weights for the liability classes are the same for all bank. It may

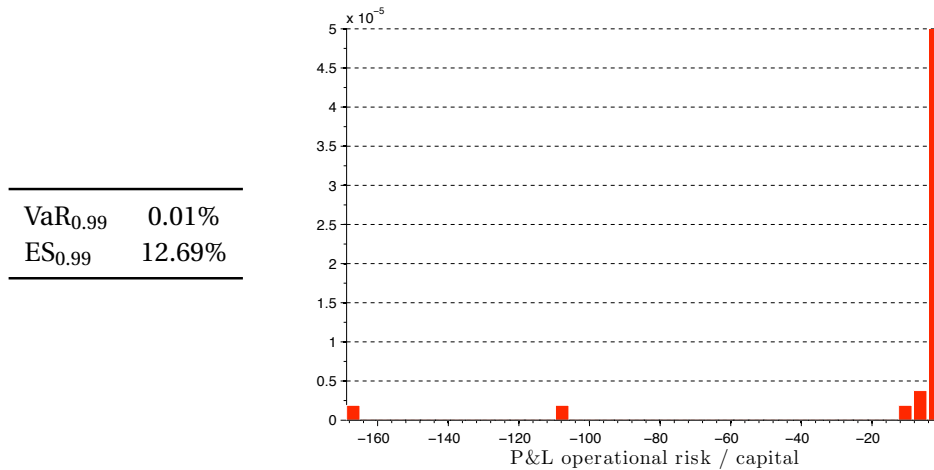


Figure 3.9: Descriptive statistics and histogram of the distribution of the operational losses of the retail bank expressed as a fraction of the initial capital. As the loss distribution is so heavy-tailed, the frequency of relatively small losses is very large, we zoomed in on the y-axis.

be more reasonable to have bank dependent weighting vectors but we keep it simple here. However, to account for the variability of funding liquidity risk between the different bank types, we assume that the investment bank has a different set of link function parameters than the retail and the universal bank. This assumption allows us to operationalize that an investment bank's funding liquidity risk is less sensitive to downward changes in the BIS ratio than retail and universal banks because fund providers expect a higher volatility due to its business model. More specifically, we use the parameters as shown in Figure 3.10.

Remark 3.5.1. Clearly, our modeling approach for A is highly stylized since we do not go into the contractual payment schedules, optionalities, etc. of the liabilities. Furthermore, we do not model the borrowing capacity of the bank, e.g., using the interest rate model plus an appropriate bank-specific credit spread (LIBOR). Here again it might be better to think of the liquidity call scenarios as a cumulative short-term total net cash outflow under stress, similar to the denominator of the LCR in Basel III.

3.5.2 Modeling proceed functions

We keep the modeling of the bank's proceed functions straightforward. In particular, we assume that we cannot liquify the loan book at all in any scenario, i.e., exclude the possibility of securization of loans. As a result, the banks can only rely on the liquidity portfolio and the trading book to generate liquidity calls. Furthermore, we assume that the proceed function for the liquidity portfolio is linear and the discount factor $\theta^{\text{lp}} \in [0, 1]$ is deterministic and equal to 0.98 so that for $0 \leq x^{\text{lp}} \leq p^{\text{lp}}$ we can write

$$G_T^{\text{lp}}(x_0^{\text{lp}}, x_1^{\text{lp}}) = x_0^{\text{lp}} + \theta^{\text{lp}} V_{1,T}^{\text{lp}} x_1^{\text{lp}}. \quad (3.12)$$

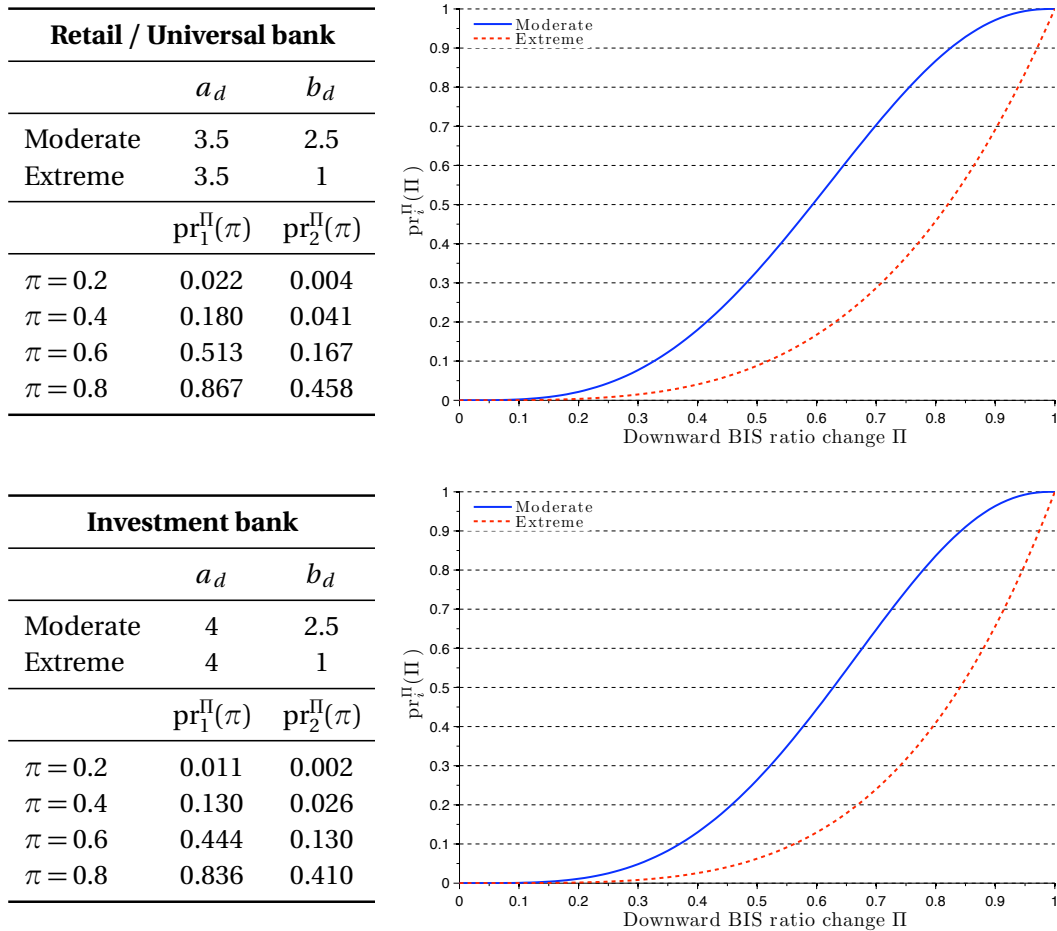


Figure 3.10: Showing the Beta CDF link function with some reasonable parameter choices for the retail and universal bank at the top and for the investment bank at the bottom. The higher the percentage downward change in the BIS ratio proxy, i.e., Π , the higher the conditional probability of a liquidity call. The link function of the moderate scenarios dominates the link function of the extreme scenario for all banks. In addition, the link functions of the investment bank are less steep than the ones of the retail and universal bank. This takes into account that fund providers of investment banks expect more volatile P&L due to the business model than fund providers of retail and universal banks.

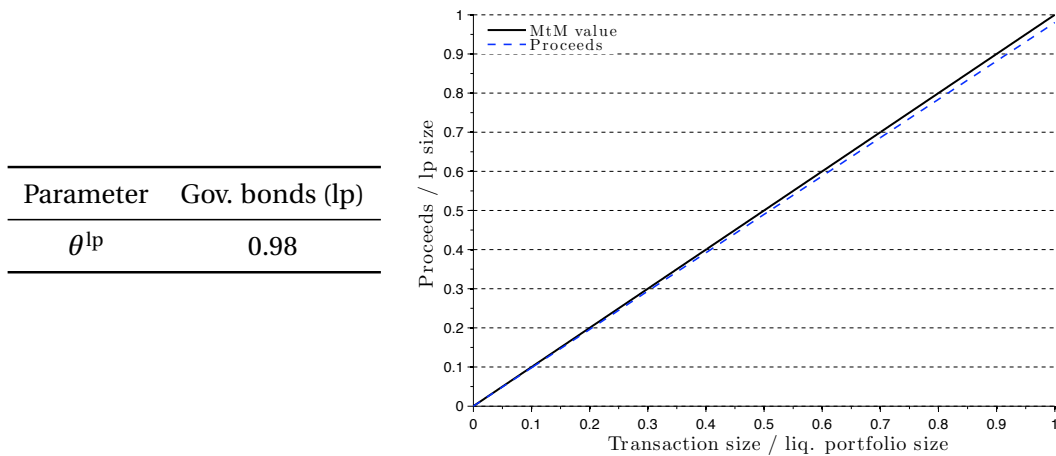


Figure 3.11: Showing the normalized proceeds of the government bonds in the liquidity portfolio as a function of the normalized transaction size, using the initial prices. Note that the liquidity costs are very small and the marginal costs are constant.

For the trading book we assume that the proceed function is of an exponential form, i.e., for a given deterministic friction parameter $\theta^{tb} \in \mathbb{R}_+$ and $0 \leq x^{tb} \leq p^{tb}$ we have that

$$G_T^{tb}(x^{tb}) = \frac{V_T^{tb}}{\theta^{tb}} (1 - e^{-\theta^{tb} x^{tb}}). \quad (3.13)$$

Note that while the friction parameters are deterministic, the proceeds are still random since the prices are. In Figure 3.11 and Figure 3.12 we show the parameters we use and plots of the asset proceeds functions using the initial prices. The parameters are chosen in an ad-hoc manner to provide reasonable results.

3.5.3 Optimal liquidity cost computations

There are several options available to compute the optimal liquidity costs per scenario:

1. Use a standard numerical search algorithm for convex optimization problems such as *fmincon* in Matlab,
2. Derive an analytical solution for our example, or
3. Use a numerical algorithm based on Lemma 2.10 on p. 18.

We choose the third option as Lemma 2.10 on p. 18 allows us to derive an efficient and robust approach to compute the optimal liquidity costs per scenario. The algorithm exploits the result of Lemma 2.10 on p. 18, which says that for a liquidation strategy to be optimal each asset position has to be liquidated up to the same μ , taking into account position bounds. The idea behind the algorithm is straightforward: we set up a grid of μ values and evaluate the optimal liquidation strategy, adjusted for any position upper bounds, per asset position at each grid point. Given the vector of optimal liquidation positions, we can compute for each grid point the corresponding α and optimal liquidity costs. Collecting these values, we can create a look-up table which can be used to give us for a given α the optimal liquidity costs. For the last step

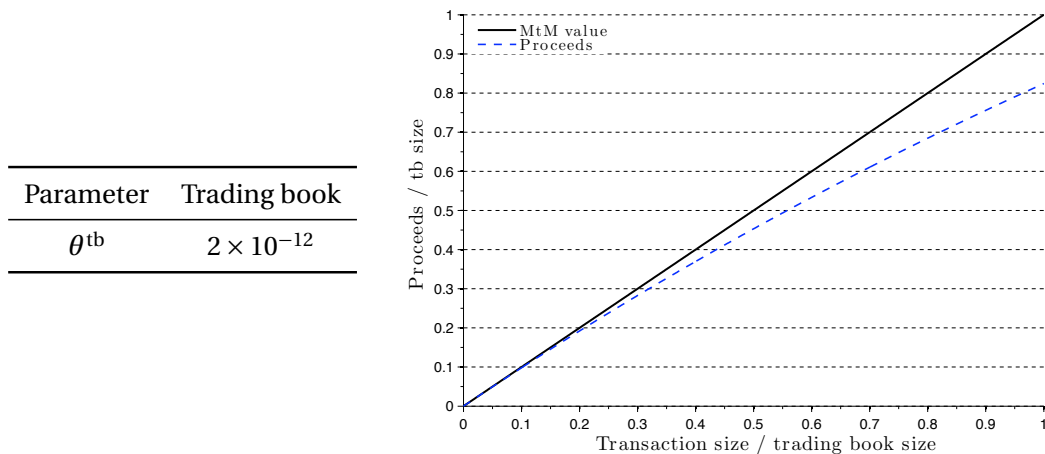


Figure 3.12: Showing the normalized proceeds of the trading book as a function of the normalized transaction size, using the initial prices. Note that the liquidity costs are initially very small but run up significantly for larger transaction sizes with increasing marginal costs.

we can use some basic interpolation methods. Our algorithm is similar in spirit to what is known as the *grid search method* but it is not quite the same, as our approach is more efficient because we can make use of the characteristics of the underlying optimization problem by exploiting Lemma 2.10 on p. 18. For that reason the common drawback of the grid search approach of exploding number of grid points does not apply. For the details of the algorithm and the corresponding Matlab code we refer the reader to the Appendix.

3.5.4 Liquidity risk characteristics of the three bank types

In order to get a feel for the liquidity risk properties of the banks, we display in Figure 3.13, Figure 3.14, and Figure 3.15 the liquidity cost profile, the normalized liquidation strategies, and the normalized optimal liquidity costs as a function of the normalized liquidity calls $\alpha/V_0(p)$, using the initial asset prices, for the retail, the universal, and the investment bank. In addition, we show an excerpt of the look-up table of our algorithm for each bank. We see that the liquidity cost profiles for all banks are very similar. Of course, this result is expected as we assumed that the asset portfolio compositions and the proceed functions are the same across the three bank types, except for the different portfolio weights. As a result, the main difference between the bank types is the point at which the banks default from illiquidity. We observe that the retail bank defaults the earliest. It defaults at liquidity calls larger than 28.66% of total assets, while the investment bank benefits from its large trading book and defaults at liquidity calls larger than 62.58% of total assets. The universal bank sits in between with a liquidity call limit of 47.70% of total assets. Meeting liquidity calls comes at the price of liquidity costs. At the default point the retail bank incurs costs of approximately 27% of the initial capital, the universal bank incurs costs of approximately 146%, and the investment bank incurs costs of approximately 280% of the initial capital.

The optimal liquidation strategy for all banks involves the following order:

1. use the available cash up to liquidity calls of a size of 1.5% of total assets for the retail and universal bank and 1% of total assets for the investment bank, as it does not result in any liquidity costs.
2. use part of the trading portfolio up until the marginal costs of the liquidity portfolio are reached, which is at liquidity calls of a size of ca. 3.4% of the balance sheet for the retail and the universal and ca. 2.9% for the investment bank.
3. completely use up the liquidity portfolio due to its constant marginal liquidity costs (linear proceed function), and finally
4. liquidate the rest of the trading book until the critical liquidity call level is reached, incurring sharply increasing marginal costs (exponential proceed function).

In Figure 3.16 and Figure 3.17 we show the liquidity cost profiles and the optimal liquidity costs for the three bank types again, but this time using the 99.99% quantiles of the marginal price distributions. As expected, the shape of the profiles is the same due to the constant friction parameters of the proceed functions. However, the critical liquidity call values for each bank are significantly lower, i.e., the banks would default at lower liquidity calls. The retail bank now barely survives a liquidity call of size 26.0%, while before it was 28.66%. The universal bank survives a liquidity call of size 42%, while before it was 47.70%. And the investment bank now survives a liquidity call of size 54%, while before it was 62.58%.

3.6 Risk aggregation

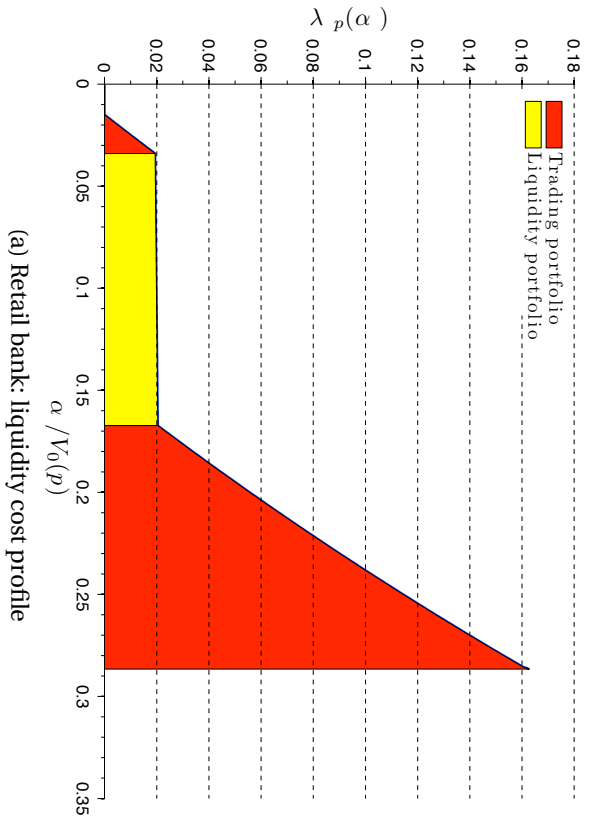
So far we only specified the marginal loss distributions. However, for the computation of (L-)EC and (L-)RAROC we need the joint probability distribution to determine the overall P&L distribution of the three bank types. There are typically two approaches used to solve the aggregation issue: (1) the hybrid approach and (2) copula approach. In the hybrid approach we actually circumvent the problem of deriving the joint probability distribution and just use a proxy for finding the overall EC from the stand-alone ECs. The copula approach is more satisfying as it allows us to specify the joint probability distribution. However, it is also more complicated than the hybrid approach.

3.6.1 Hybrid approach

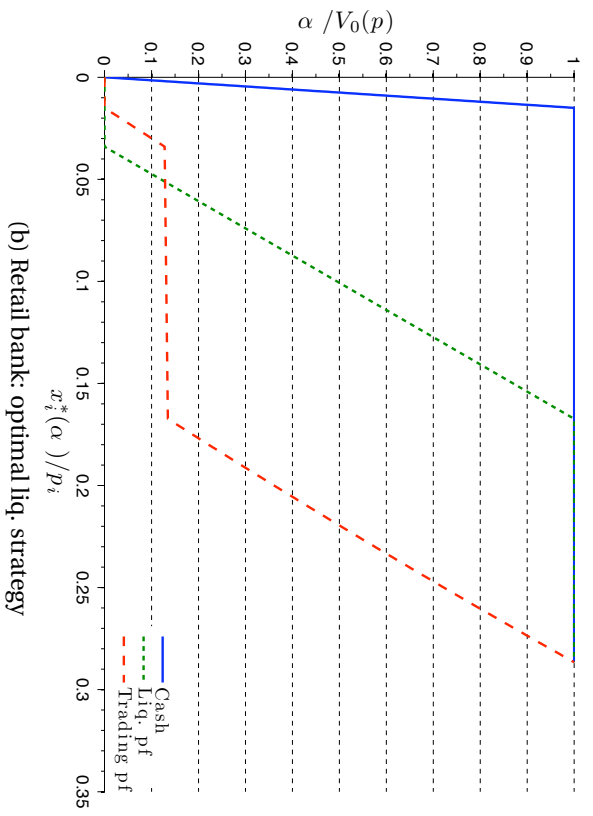
Let us denote the stand-alone EC of the liquidity portfolio by $EC(p^{\text{lp}})$, the stand-alone EC of the loan book by $EC(p^{\text{lb}})$, and the stand-alone EC of the trading book by $EC(p^{\text{tb}})$. Then under the hybrid approach the overall EC of the bank $EC(\hat{p})$ is given by

$$EC(\hat{p}) = \sqrt{\sum_{i \in J} \sum_{j \in J} EC(p^i) EC(p^j) \rho^{i,j}},$$

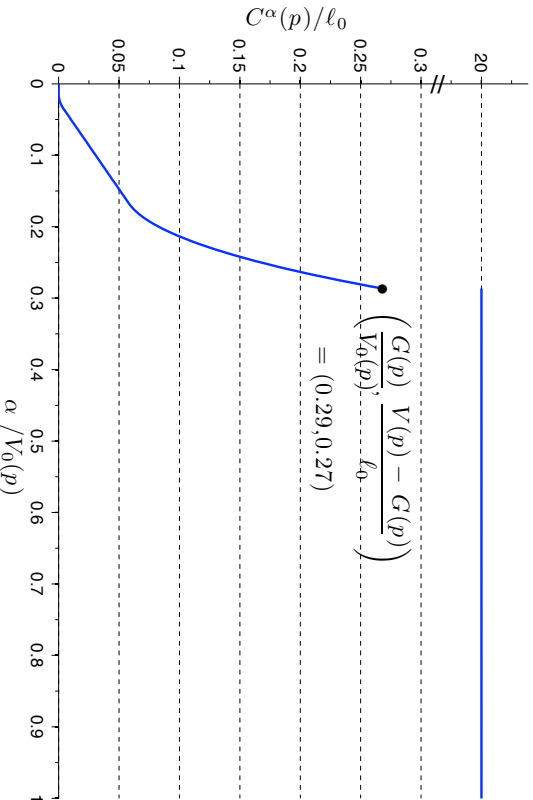
where $J = \{\text{lp}, \text{lb}, \text{tb}\}$ and $\rho^{i,j}$ stands for the correlation between i and j . The approach is generally less conservative than simply adding up the stand-alone ECs, i.e., $\rho^{i,j} = 1$



(a) Retail bank: liquidity cost profile



(b) Retail bank: optimal liq. strategy

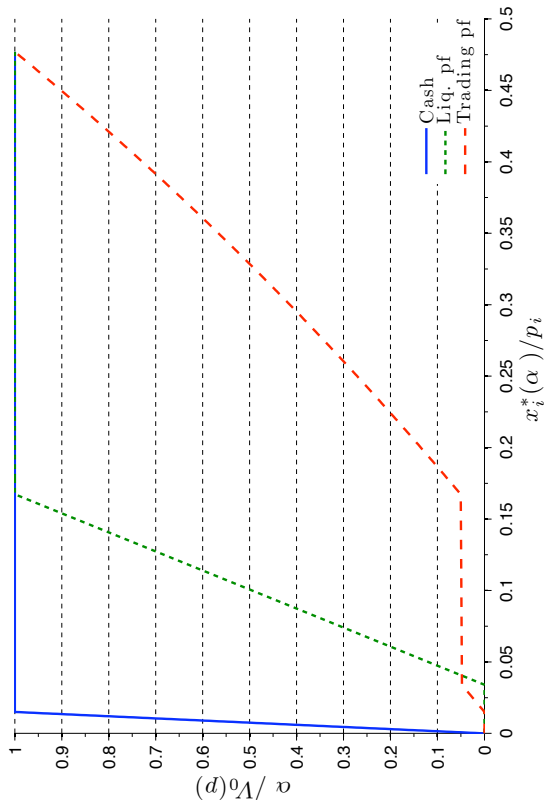


(c) Retail bank: optimal liq. costs

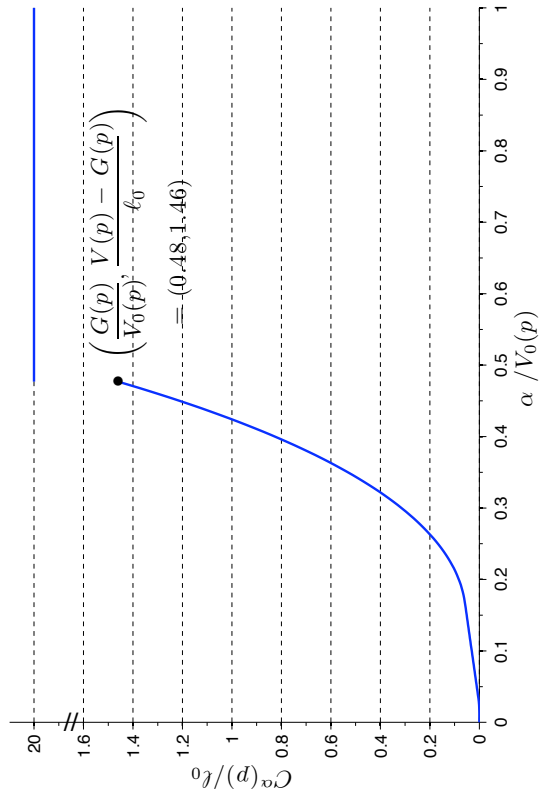
$\mu_p(\alpha)$	x_{lp}^*/p^{lp}	x_{tb}^*/p^{tb}	$\alpha/V_0(p)$	$C^\alpha(p)/l_0$
1.0000	0.0000	0.0000	0.0150	0.0000
0.9990	0.0000	0.0067	0.0160	0.0000
⋮	⋮	⋮	⋮	⋮
0.9810	0.0000	0.1279	0.0340	0.0037
0.9800	1.0000	0.1347	0.1673	0.0581
0.9790	1.0000	0.1415	0.1683	0.0585
⋮	⋮	⋮	⋮	⋮
0.8950	1.0000	0.7395	0.2523	0.1726
⋮	⋮	⋮	⋮	⋮
0.8610	1.0000	0.9977	0.2863	0.2672
0.8600	1.0000	1.0000	0.2866	0.2682

(d) Retail bank: look-up table

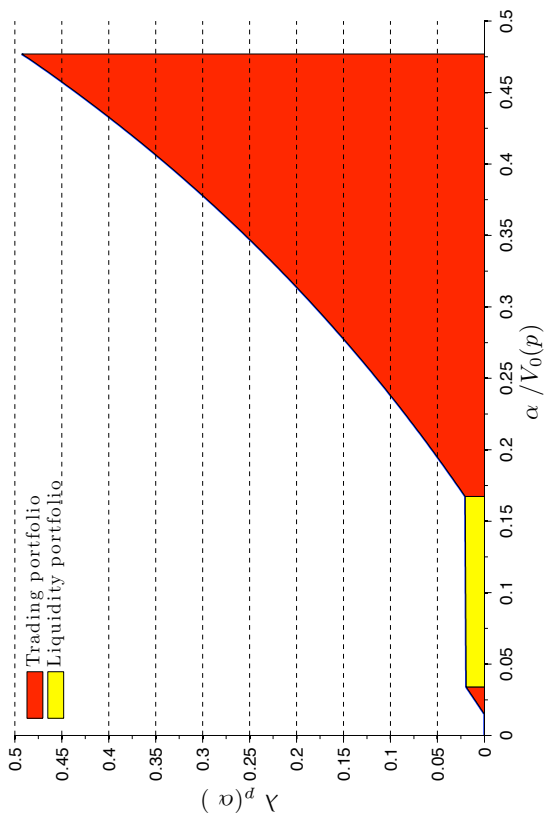
Figure 3.13: The liquidity cost profile, the normalized liquidation strategies, and the normalized optimal liquidation strategies as a function of the normalized liquidity calls $\alpha/V_0(p)$ for the retail bank.



(a) Universal bank: liquidity cost profile

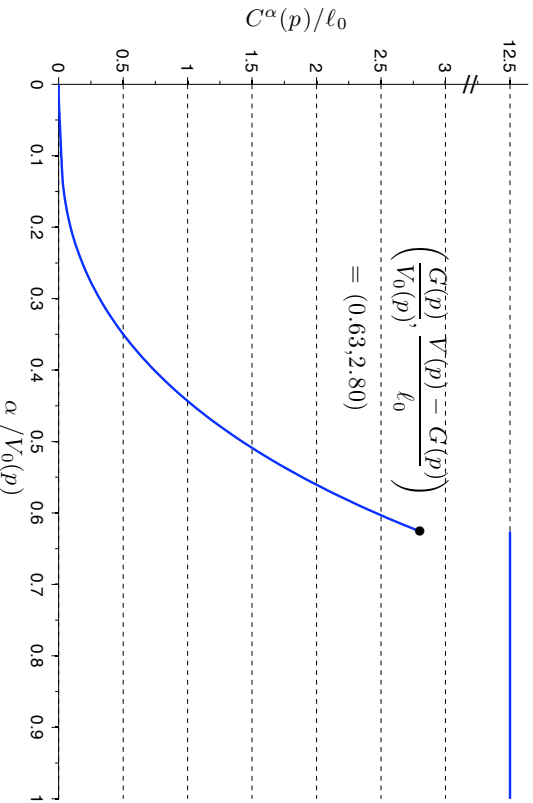
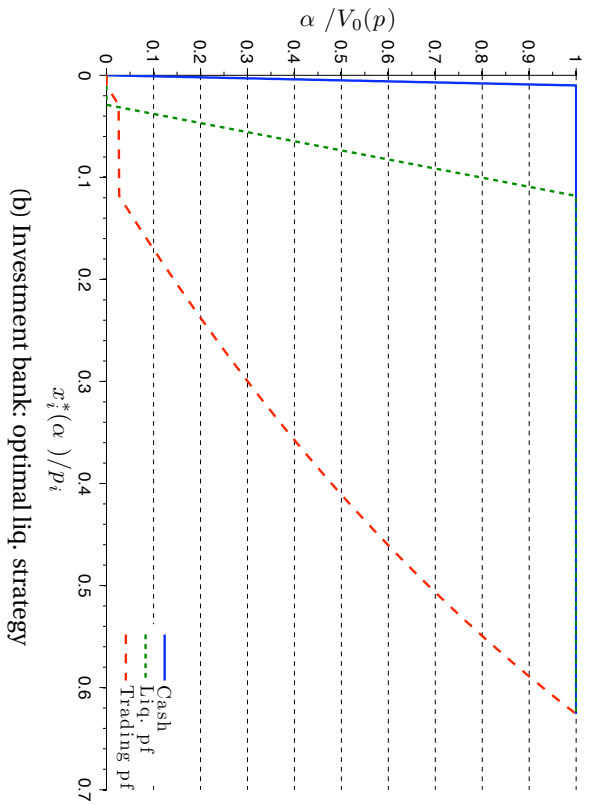
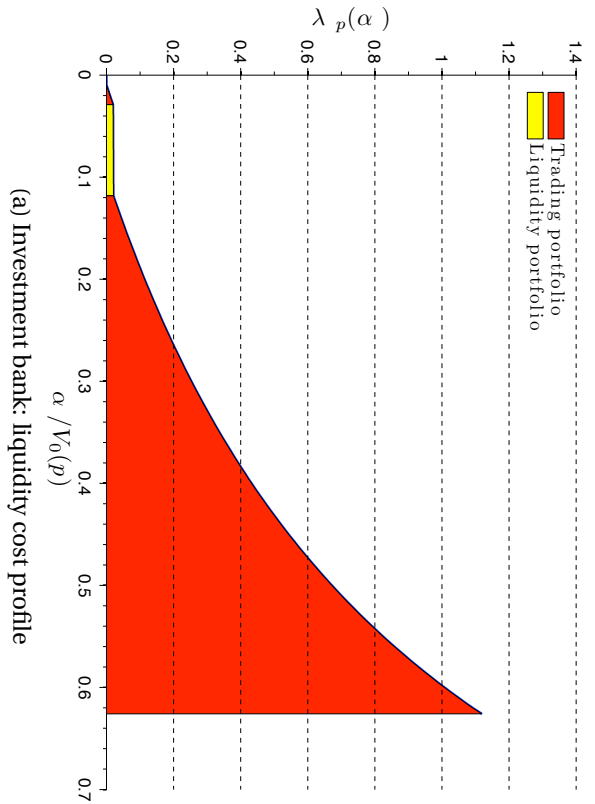


(b) Universal bank: optimal liq. costs



(c) Universal bank: look-up table

Figure 3.14: The liquidity cost profile, the normalized liquidation strategies, and the normalized optimal liquidation strategies, and the normalized optimal liquidity costs as a function of the normalized liquidity calls $\alpha/V_0(p)$ for the universal bank.



(c) Investment bank: optimal liq. costs

$\mu_p(\alpha)$	x_{lp}^*/p_{lp}	x_{tb}^*/p_{tb}	$\alpha/V_0(p)$	$C^\alpha(p)/\ell_0$
1.0000	0.0000	0.0000	0.0100	0.0000
0.9990	0.0000	0.0013	0.0110	0.0000
⋮	⋮	⋮	⋮	⋮
0.9810	0.0000	0.0256	0.0290	0.0023
0.9800	1.0000	0.0269	0.1182	0.0250
0.9790	1.0000	0.0283	0.1192	0.0253
⋮	⋮	⋮	⋮	⋮
0.6240	1.0000	0.6288	0.4742	1.2176
⋮	⋮	⋮	⋮	⋮
0.4730	1.0000	0.9982	0.6252	2.7932
0.4720	1.0000	1.0000	0.6258	2.8021

(d) Investment bank: look-up table

Figure 3.15: The liquidity cost profile, the normalized liquidation strategies, and the normalized optimal liquidity costs as a function of the normalized liquidity calls $\alpha/V_0(p)$ for the investment bank.

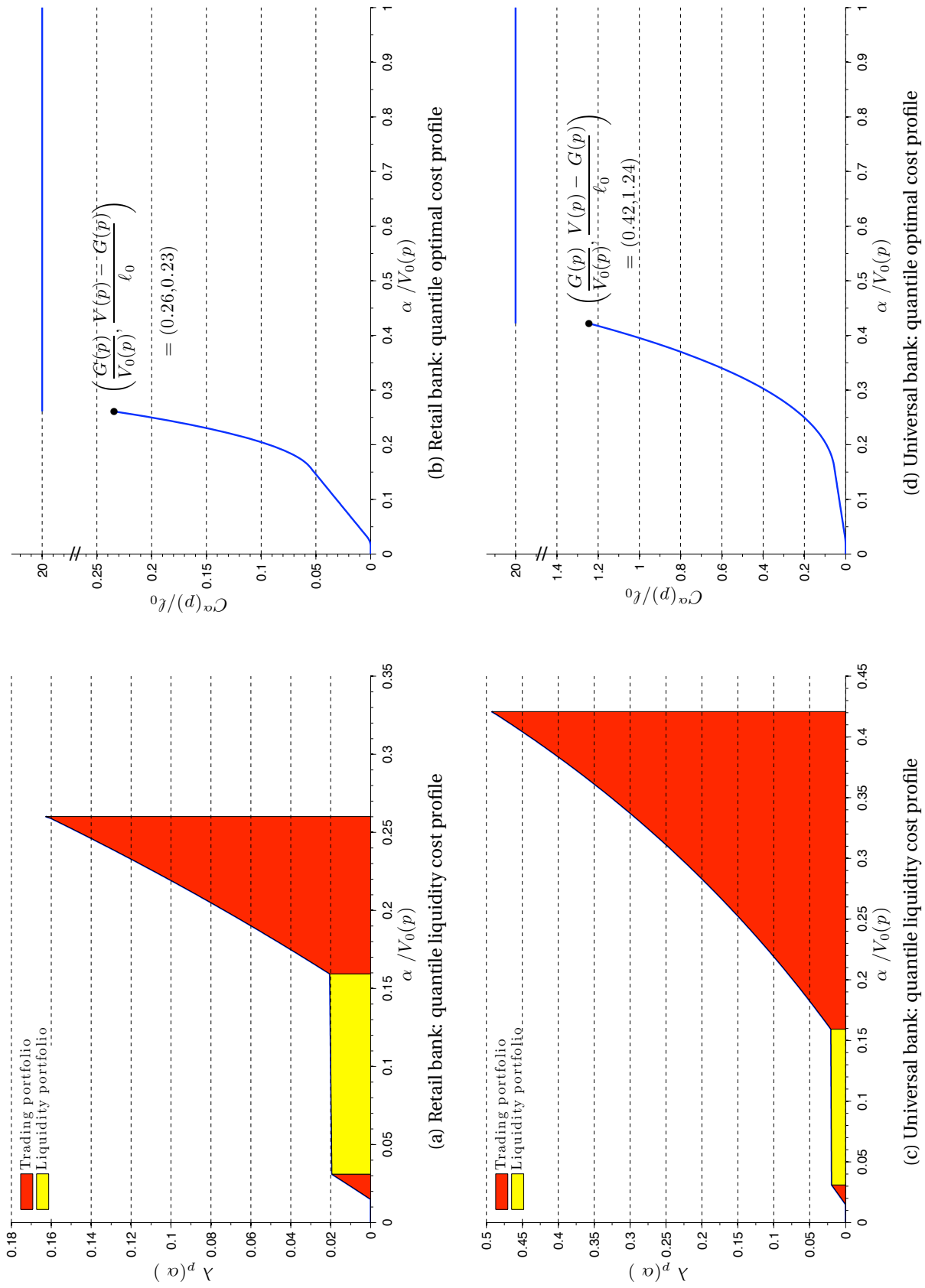
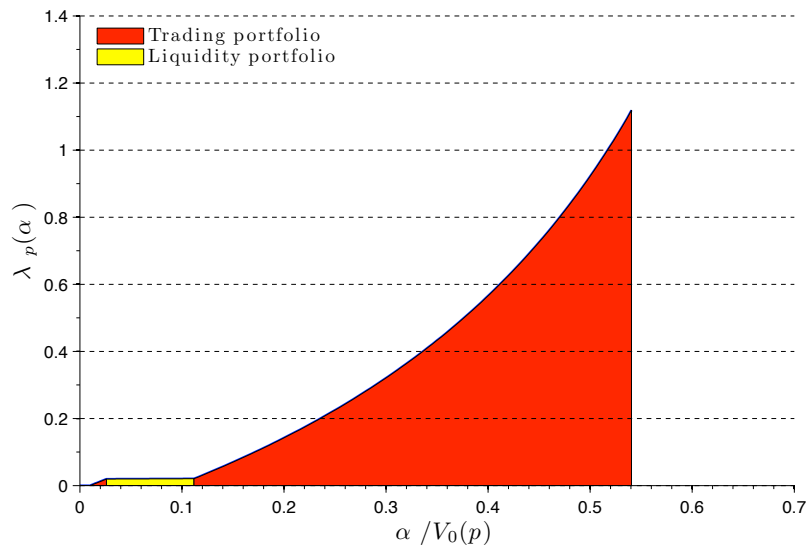
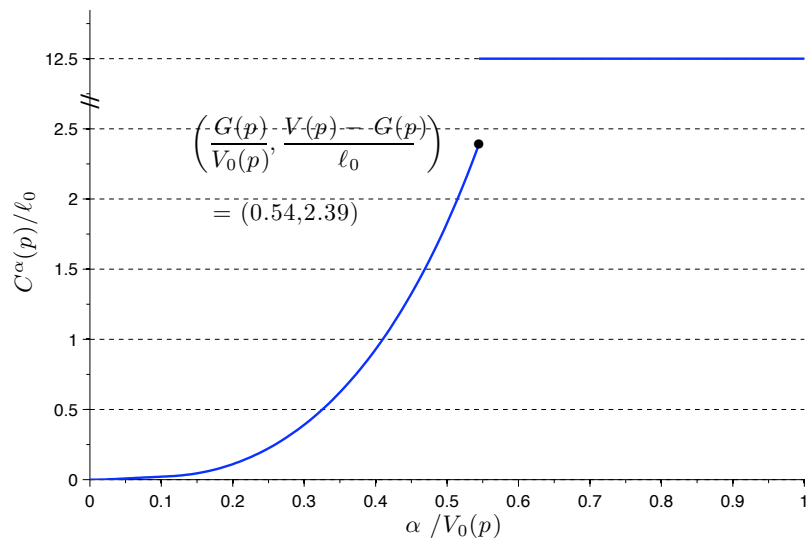


Figure 3.16: The liquidity cost and optimal liquidity cost profiles as a function of the normalized liquidity calls $\alpha / V_0(p)$ for the retail and the universal banks, using the 99.99% price quantiles.



(a) Investment bank: quantile liquidity cost profile



(b) Investment bank: quantile optimal cost profile

Figure 3.17: The liquidity cost and optimal liquidity cost profiles as a function of the normalized liquidity calls $\alpha/V_0(p)$ for the investment bank, using the 99.99% price quantiles.

	liquidity pf	trading/market risk	loans/credit risk	Op-risk
liquidity pf	1	0.3	0.3	0.2
trading/market risk	0.3	1	0.5	0.2
loans/credit risk	0.3	0.5	1	0.2
Op-risk	0.2	0.2	0.2	1

Table 3.3: Correlation parameters between different risk types are based on Rosenberg and Schuermann (2006).

for all i and j , as is required for the regulatory capital under Basel II.¹³ While the hybrid approach is often used in practice, we will use on the copula approach in this chapter because we need the full joint probability distribution to compute the liquidity adjustment.

3.6.2 Copula approach

Under the copula approach we complete the joint probability model by joining the marginal distributions with a Student's t copula with ν degree of freedom and correlation matrix P . We use the Student's t copula because it can capture “tail dependence”, in contrast to a normal copula, which has tail independence (McNeil et al., 2005). More formally, given the marginal distributions functions $F_{X(p^{\text{lp}})}(x_1)$, $F_{X(p^{\text{tb}})}(x_2)$, $F_{X(p^{\text{lb}})}(x_3)$, and $F_{L^{\text{Op}}}(x_4)$, we assume that the joint probability distribution for the bank's P&L is given by

$$\begin{aligned} F(x_1, x_2, x_3, x_4) &:= \mathbb{P}(X(p^{\text{lp}}) \leq x_1, X(p^{\text{tb}}) \leq x_2, X(p^{\text{lb}}) \leq x_3, L^{\text{Op}} \leq x_4) \\ &= C_{\nu, P}^t(F_{X(p^{\text{lp}})}(x_1), F_{X(p^{\text{tb}})}(x_2), F_{X(p^{\text{lb}})}(x_3), F_{L^{\text{Op}}}(x_4)), \end{aligned}$$

with $C_{\nu, P}^t : [0, 1]^d \rightarrow [0, 1]$ and

$$C_{\nu, P}^t(u_1, \dots, u_d) := \int_{-\infty}^{t_{\nu}^{-1}(u_1)} \dots \int_{-\infty}^{t_{\nu}^{-1}(u_d)} \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^d |P|}} \left(1 + \frac{\mathbf{x}'P^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+d}{2}} dx,$$

where P is the correlation matrix implied by the dispersion matrix Σ and t_{ν}^{-1} denotes the quantile function of a standard univariate Student's t distribution with ν degrees of freedom. The parameter ν determines the extent of tail dependence. In the literature, ν ranges from three to eight for strong tail dependence (Demarta and McNeil, 2005), so we take a midpoint: $\nu = 5.5$. We use the correlation parameters in Table 3.3. The estimates are based on the benchmark correlations used in Rosenberg and Schuermann (2006). The authors take the midpoint inter-risk correlation estimates from two academic

¹³Note that adding up is generally not the worst-case scenario (see Fallacy 3 on p. 251 in McNeil et al., 2005)

studies and one industry study.¹⁴ Consequently, P is given by

$$P = \begin{bmatrix} 1 & 0.3 & 0.3 & 0.2 \\ 0.3 & 1 & 0.5 & 0.2 \\ 0.3 & 0.5 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}.$$

It is rather straightforward to sample from the joint probability distribution, given we have specified the Student's t copula and the correlations. We refer the reader to the Appendix for further details regarding the sampling procedure.

3.7 Results

3.7.1 Bank type and liquidity risk

Without liquidity risk the ordering of the riskiness is in line with what one would expect from the description of the bank types. The investment bank is the riskiest, the commercial is the safest, and the universal bank takes the middle position. Note that we express the results as a fraction of the initial capital and as the investment bank has higher initial capital risk measure, we first need to convert the results to compare them to the other two bank types. The investment bank's VaR of 79.08% becomes 126.53% ($0.7908\phi_{\text{equity}}^{\text{inv}}/\phi_{\text{equity}}^{\text{ret}}$) and the ES of 95.22% becomes 152.35% ($0.9522\phi_{\text{equity}}^{\text{inv}}/\phi_{\text{equity}}^{\text{ret}}$) expressed as a fraction of the retail and universal bank's initial capital. All three banks pass the VaR test of capital adequacy by a reasonable margin. With regard to the ES test only the universal bank fails to pass it. In case ES is used for an capital adequacy analysis, the universal bank falls short by 8.87% and the management would need to engage in corrective actions. The probability of default (PD) of all three banks is as expected well below the confidence level of 1%.¹⁵ In line with the ES result the PD of the universal is the highest with 0.48%. Note that the PD of the investment bank is lower only because of the higher initial capital. In terms of risk-adjusted performance, the three banks are pretty similar for both RAROC with VaR and RAROC with ES, with the investment bank having the lowest ratio of the three bank types by a small margin.

The results become more interesting with the inclusion of liquidity risk. As expected the inclusion of liquidity risk increase the riskiness and decreases the risk-adjusted performance across all three bank types. Liquidity risks increases the VaR by 17.03%, 35.85%, and 121.61% for the retail, universal, and investment bank, respectively. That would mean a failure to comply with a capital adequacy test for the universal and investment bank. The latter is also reflected in the sharply increased PDs of the two bank types. The universal bank now has a PD of 2.56% and the investment bank a PD of 5.4%. The influence of liquidity risk has a similar impact on ES as it has on the VaR,

¹⁴Note that the linear correlation coefficients are not preserved under the nonlinear transformation to the joint model.

¹⁵Note, however, that the PDs are still high compared to the average cumulative default rates of Moody's (see, for instance, p. 292 in Hull (2010)), the three banks have a PD of around Baa (0.17%). However, these estimates do not take the more recent events of the Subprime crisis into account.

	Retail	Universal	Investment
$\text{VaR}_{0.99}/\ell_0$	68.07%	86.67%	79.08%
$\text{ES}_{0.99}/\ell_0$	88.72%	108.87%	95.22%
$\text{RAROC}_{\text{VaR}}$	13.65%	14.04%	12.50%
RAROC_{ES}	10.47%	11.18%	10.38%
PD_X^\dagger	0.14%	0.48%	0.21%
$\text{L-VaR}_{0.99}/\ell_0$	85.10%	122.52%	200.69%
$\text{L-ES}_{0.99}/\ell_0$	935.40%	143.66%	218.84%
$\text{L-RAROC}_{\text{VaR}}$	0.59%	8.25%	1.77%
$\text{L-RAROC}_{\text{ES}}$	0.05%	7.03%	1.63%
$\text{PD}_{X^A}^{\dagger\dagger}$	0.67%	2.56%	5.40%

$$\dagger \text{PD}_X := \mathbb{P}\{X(\hat{p}) < -\ell_0\},$$

$$\dagger\dagger \text{PD}_{X^A} := \mathbb{P}\{X^A(\hat{p}) < -\ell_0\}.$$

Table 3.4: Overview of the statistics of the overall P&L with and without liquidity risk, i.e., X and X^A , for the three bank types (simulation trials = 250,000). The addition of liquidity risk increases the riskiness and decreases the risk-adjusted performances markedly. Using L-VaR, the universal and the investment bank fail the capital adequacy test, as the initial capital levels fail to cover the capital requirements, while the retail passes. Using L-ES, all three banks fail the adequacy test. Note that the high L-ES value of the retail bank can be explained by the fact that the retail bank defaults in some scenarios due to illiquidity and hence suffers from the 100% rule. Also note that we do not claim that the PDs values after the liquidity risk adjustment are meaningful beyond our stylized illustration.

with the exception for the retail bank. The retail bank faces an increase of 846.68% in its ES number as a result of liquidity risk. This large increase can be explained by the 100% rule that goes into effect in scenarios where a bank cannot meet the liquidity call and as a result defaults. Interestingly, these default states are only reflected in the histogram visually (see Figure 3.18 on p. 93) and numerically in the outcome of L-ES. However, this result makes sense as the 100% rule is far in the tail and at our confidence level, L-VaR does not take it into account, as can be induced from the fact that the PD with liquidity risk, while increased, is still less than the confidence level: $\text{PD}_{X^A} = 0.67\% < 1\%$. Liquidity risk causes the RAROC figures to decrease steeply for all banks. However, the retail bank again stands out due to the fact that the expected P&L (numerator of RAROC), while still positive, is markedly affected by the workings of the 100% rule.

In Figure 3.21 on p. 96, Figure 3.22 on p. 97, and Figure 3.23 on p. 98 we investigate the effects of liquidity risk for the three bank types in more detail. We present the expected value, VaR, and ES of the optimal liquidity costs $C_T^A(\hat{p})$, the unconditional probabilities of the three liquidity call scenarios, and the histograms of the conditional liquidity cost distributions. We first notice that of the three banks, the retail bank has the lowest probabilities of having moderate or extreme liquidity calls. Interestingly, the universal bank has slightly higher occurrence probabilities than the investment bank. This effect can mainly be explained by the fact that the higher volatility of the BIS ratio of the investment bank is overpowered by the lesser sensitivity to ratio

changes via the beta parameters of the liquidity call function (see Figure 3.10 on p. 80). As the liquidity call size is not random, we already knew that the investment bank has the largest liquidity call size, followed by the universal bank, and then the retail bank, reflecting the riskiness of the funding structure (portfolio weights).

The histograms in the previous figures give a good overview of the impact the moderate and extreme liquidity call scenarios have on the incurred liquidity costs. In all three cases, there is a significant difference between the magnitude of the liquidity costs between the moderate and the extreme liquidity call scenario. This is most apparent for the retail bank, as the moderate scenario only leads to relatively small liquidity costs ($\text{VaR}_{0.99}(C_T^A(\hat{p}))/\ell_0 = 8.6\%$), while the extreme scenario leads to the bank's default and hence the application of the 100%, whenever it occurs. This result was to be expected as the extreme scenario of the retail bank leads to a liquidity call of 32% of the total assets and we know from Figure 3.16 on p. 87 that the default barrier, using the 99.99% quantile of the marginal price distributions, is at around 26% of the total assets. Similarly, we could have expected that the universal and investment bank would not default by illiquidity states. Note also, that for the retail bank the probability of the extreme scenario occurring is less than the confidence level of VaR: $\mathbb{P}\{\text{extreme scenario}\} = 0.44\% < 1\%$. This fact explains more directly why the L-VaR is not sensitive to the default by illiquidity states and hence the 100%.¹⁶ The liquidity costs characteristics of the investment bank are quite different from that of the retail bank. In both the moderate and the extreme scenario the liquidity costs are very large, which is reflected in the VaR and ES of the liquidity costs of 118.81% and 135.57%, respectively. However, despite the large liquidity call size of 44.98% and 48.98% of the total assets, the investment bank does not come close to its default barrier, even in the extreme scenarios as it is above 50% in those cases (cf. Figure 3.17 on p. 88). The picture of the universal bank is similar to that of the investment bank, except that the liquidity costs are significantly smaller in general.

3.7.2 Impact of bank size

In Theorem 2.22 on p. 33 and Theorem 2.26 on p. 38 we have confirmed the common intuition that size does matter in the face of liquidity risk when it comes to the determination of capital requirements. Without liquidity risk position size does not matter since linearity in positions and positive homogeneity of the risk measure ensures that for $\lambda \geq 0$ it holds that

$$\text{EC}(\lambda p) = \lambda \text{EC}(p)$$

In contrast, we could show that if we use a positively (super)homogenous underlying risk measure and a convex A we have for $\lambda \geq 1$ and $\hat{p} \in \hat{\mathcal{P}}$ that

$$\text{L-EC}(\lambda \hat{p}) \geq \lambda \text{L-EC}(\hat{p}).$$

¹⁶Recall Equation 2.10 on p. 29 and the general discussion of the sensitivity of risk measures to the default by illiquidity states.

Retail bank

$V_0^{lp}(p^{lp})$	€ 75bn	ℓ_0	€ 50bn
$V_0^{lb}(p^{lb})$	€ 350bn	$L_0^{dep}(\ell^{dep})$	€ 300bn
$V_0^{tb}(p^{tb})$	€ 75bn	$L_0^{st}(\ell^{st})$	€ 75bn
	€ 500bn	$L_0^{lr}(\ell^{lr})$	€ 75bn
			€ 500bn

Parameter	X	X^A	Δ
$Var_{0,99}/\ell_0$	68.07%	85.10%	+17.03%
$ES_{0,99}/\ell_0$	88.72%	935.40%	+846.68%
PD	0.14%	0.67%	+0.53%
$RAROC_{Var}$	13.65%	0.59%	-13.06%
$RAROC_{ES}$	10.47%	0.05%	-10.42%

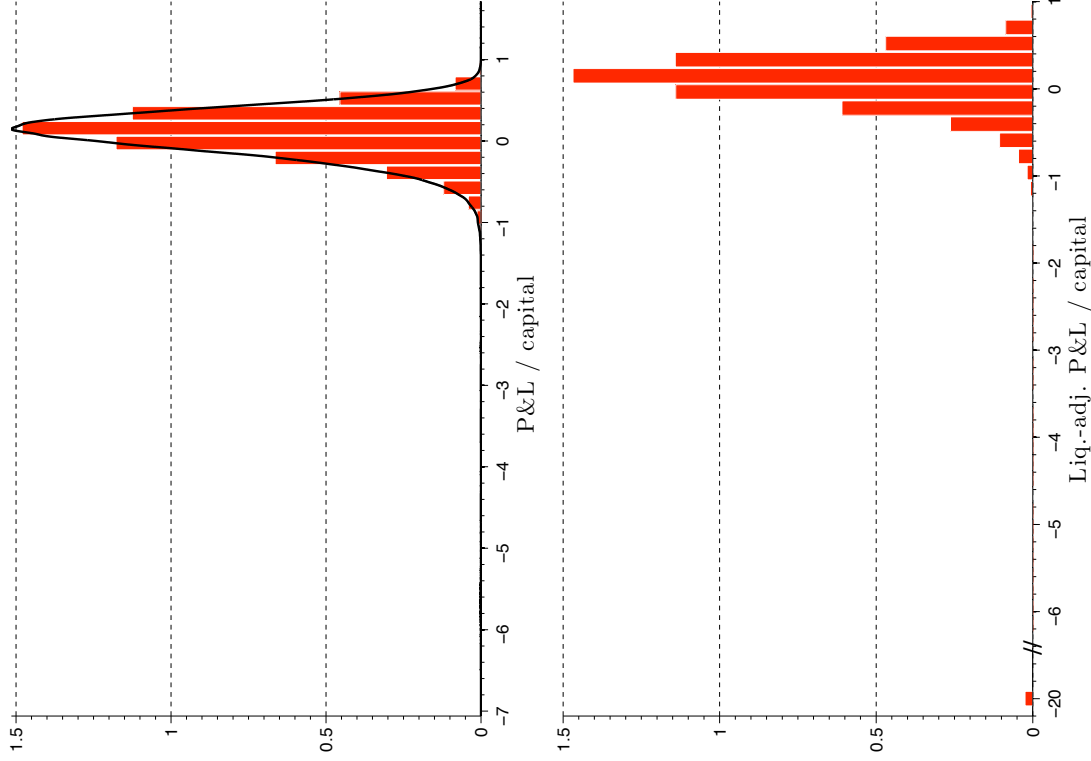


Figure 3.18: Descriptive statistics (left), the histogram of the standard (upper right) overall P&L distributions for retail bank, and the liquidity-adjusted (lower right) overall P&L distributions for retail bank.

Universal bank			
$V_0^{\text{lb}}(p^{\text{lb}})$	€ 75bn	ℓ_0	€ 50bn
$V_0^{\text{lb}}(p^{\text{lb}})$	€ 350bn	$L_0^{\text{dep}}(\ell^{\text{dep}})$	€ 300bn
$V_0^{\text{lb}}(p^{\text{tb}})$	€ 75bn	$L_0^{\text{st}}(\ell^{\text{st}})$	€ 75bn
		$L_0^{\text{lt}}(\ell^{\text{lt}})$	€ 75bn
	€ 500bn		€ 500bn

Parameter	X	X^A	Δ
$\text{VaR}_{0.99}/\ell_0$	86.67%	122.52%	+35.85%
$\text{ES}_{0.99}/\ell_0$	108.87%	143.66%	+34.79%
PD	0.48%	2.56%	+2.08%
$\text{RAROC}_{\text{var}}$	14.04%	8.25%	-5.79%
RAROC_{ES}	11.18%	7.03%	-4.15%

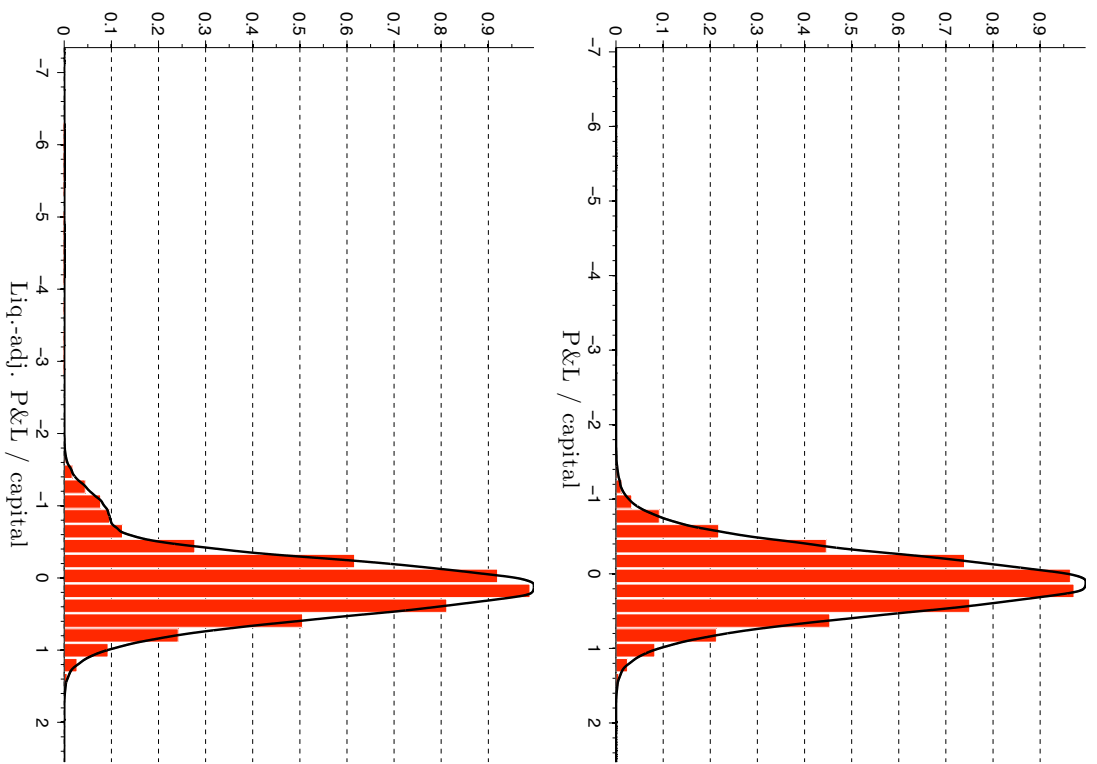


Figure 3.19: Descriptive statistics (left), the histogram of the standard (upper right), and the liquidity-adjusted (lower right) overall P&L distributions for universal bank.

Investment bank

$V_0^{\text{ip}}(p^{\text{ip}})$	€ 75bn	ℓ_0	€ 50bn
$V_0^{\text{lb}}(p^{\text{lb}})$	€ 350bn	$L_0^{\text{dep}}(\ell^{\text{dep}})$	€ 300bn
$V_0^{\text{tb}}(p^{\text{tb}})$	€ 75bn	$L_0^{\text{st}}(\ell^{\text{st}})$	€ 75bn
		$L_0^{\text{lt}}(\ell^{\text{lt}})$	€ 75bn
	€ 500bn		€ 500bn

Parameter	X	X^A	Δ
$\text{VaR}_{0.99}/\ell_0$	79.08%	200.69%	+121.61%
$\text{ES}_{0.99}/\ell_0$	95.22%	218.84%	+123.62%
PD	0.21%	5.4%	+5.19%
$\text{RAROC}_{\text{VaR}}$	12.50%	1.77%	-10.73%
RAROC_{ES}	10.38%	1.63%	-8.75%

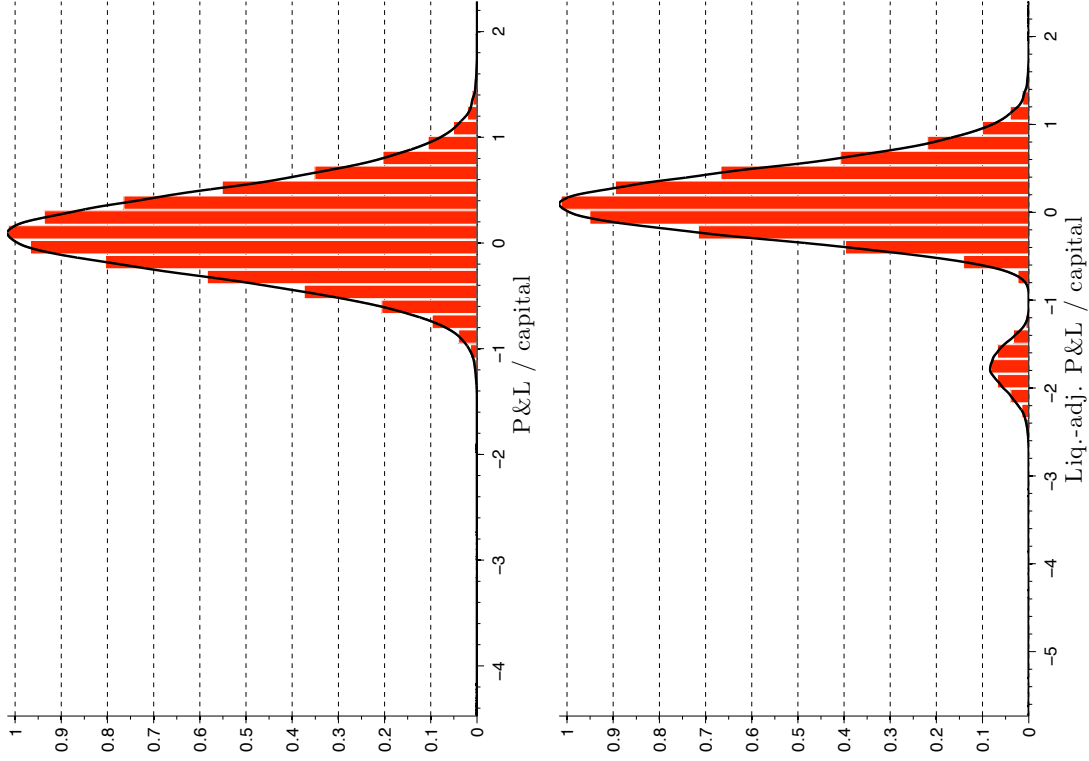
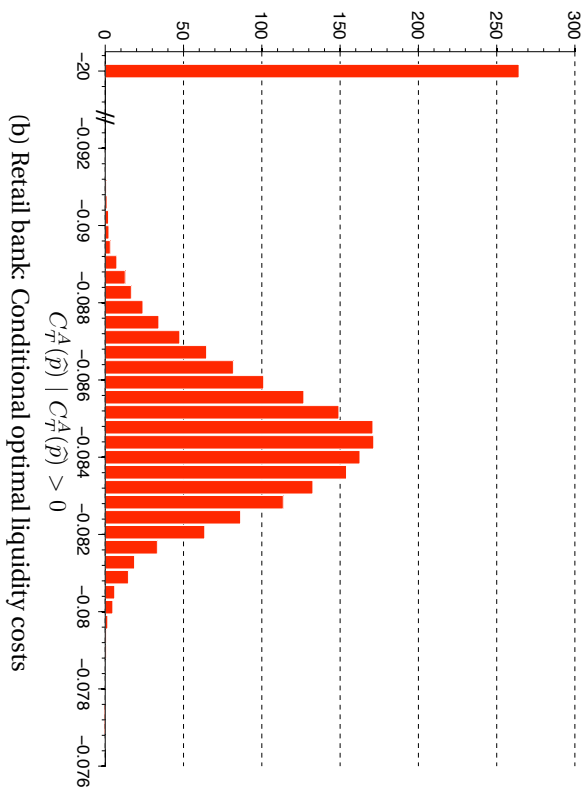


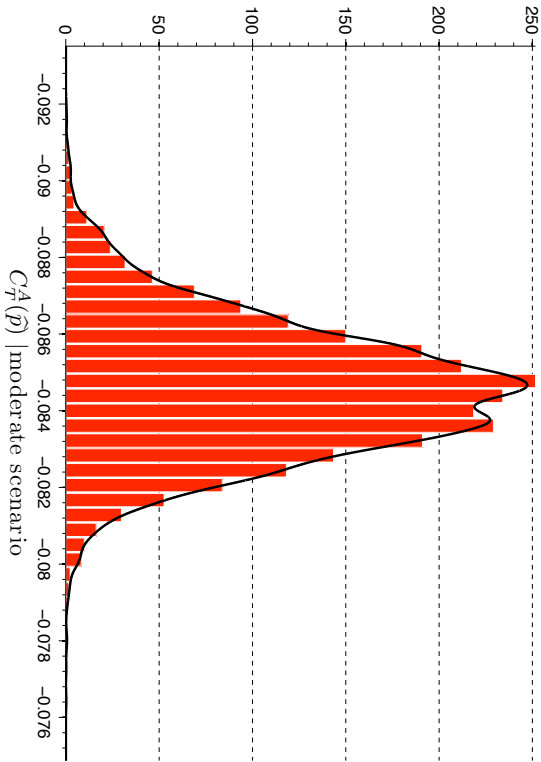
Figure 3.20: Descriptive statistics (left), the histogram of the standard (upper right), and the liquidity-adjusted (lower right) overall P&L distributions for investment bank.

Optimal liquidity costs	
Expected value	9.00%
$VaR_{0.99}/\ell_0$	8.60%
$ES_{0.99}/\ell_0$	878.50%
Liquidity calls	
$\alpha(\omega)/V_0(p)$	$\mathbb{P}\{\omega\}$
no call	0.00%
moderate	20.00%
extreme	32.00%
	0.44%

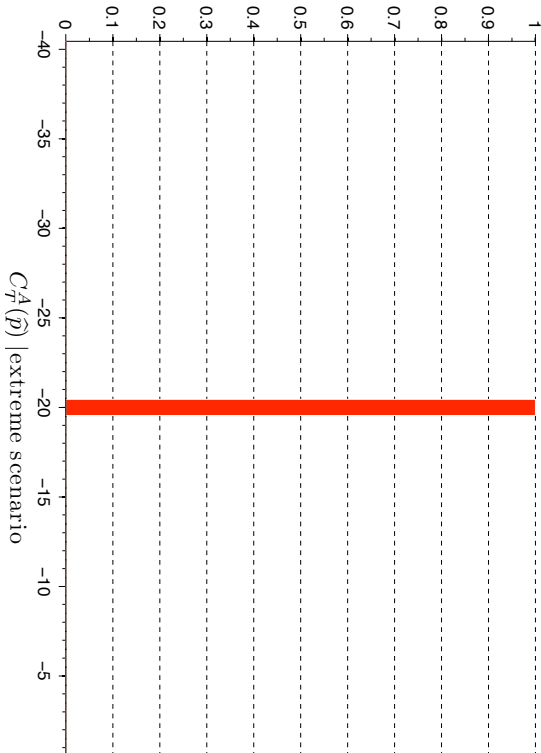
(a) Retail bank: Statistics



(b) Retail bank: Conditional optimal liquidity costs



(c) Retail bank: Conditional optimal liq. costs under moderate scenario



(d) Retail bank: Conditional optimal liq. costs under extreme scenario

Figure 3.21: Overview of the statistics of the optimal liquidity costs $C_T^A(p)$, the three liquidity call scenarios and the corresponding unconditional probabilities, and the histograms of the conditional distributions for the retail bank (simulation trials = 250,000). In all three figures the optimal liquidity costs are expressed in units of initial capital.

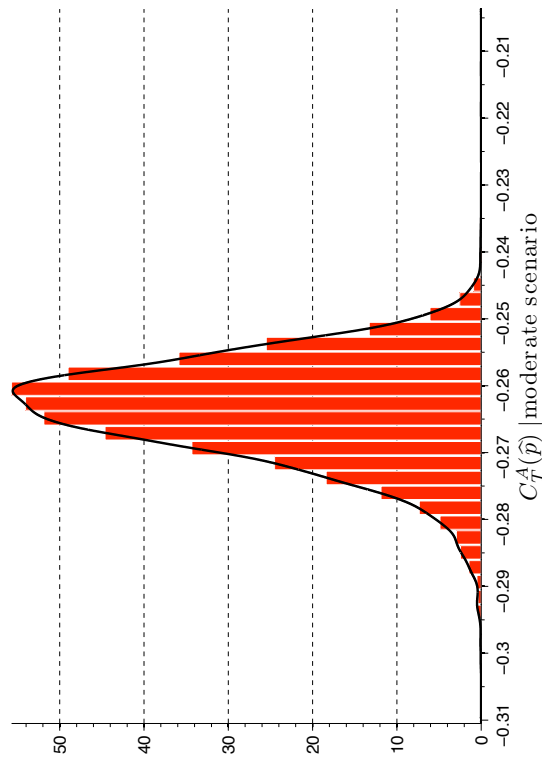
Optimal liquidity costs

Expected value	2.07%
$\text{VaR}_{0.99}/\ell_0$	27.72%
$\text{ES}_{0.99}/\ell_0$	60.18%

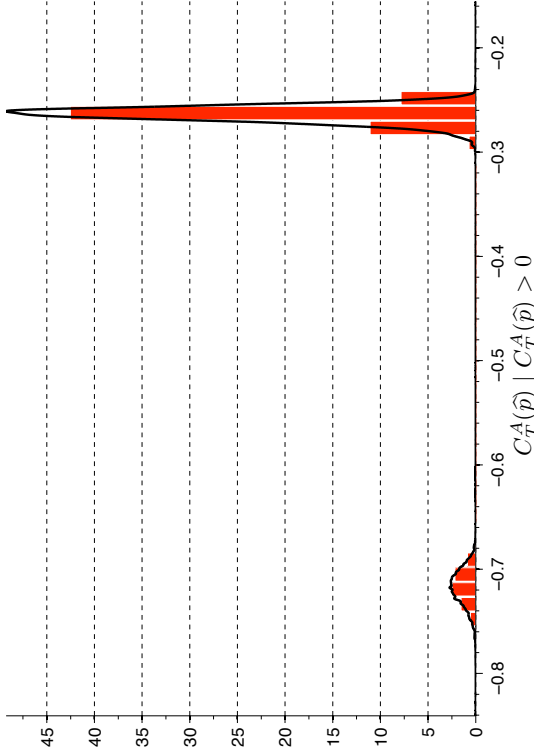
Liquidity calls

	$\alpha(\omega)/V_0(p)$	$\mathbb{P}\{\omega\}$
no call	0.00%	93.43%
moderate	28.10%	5.84%
extreme	37.70%	0.74%

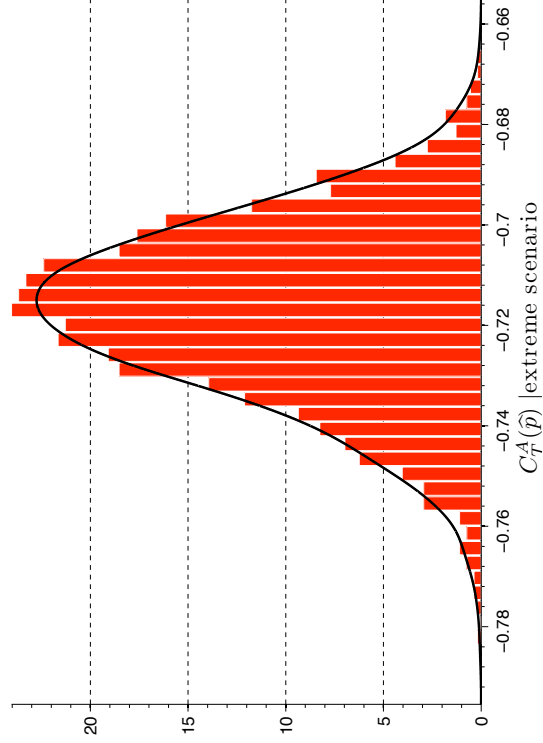
(a) Universal bank: Statistics



(c) Universal bank: Cond. optimal liq. costs under moderate scenario



(b) Universal bank: Conditional optimal liquidity costs

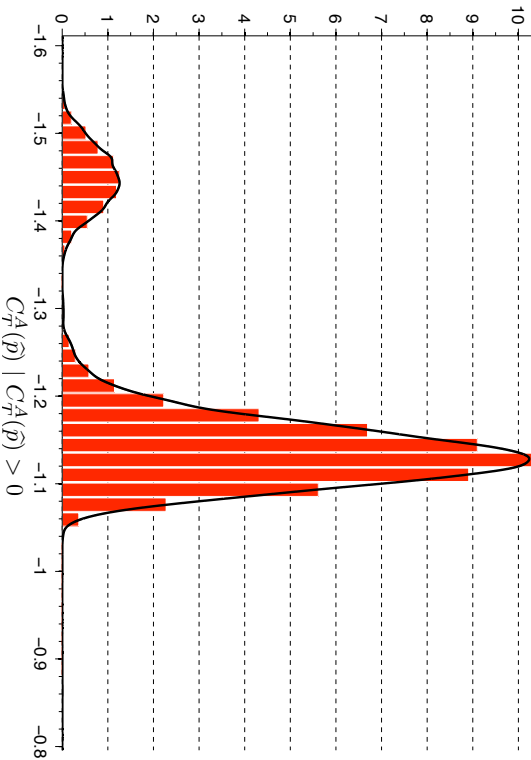


(d) Universal bank: Conditional optimal liq. costs under extreme scenario

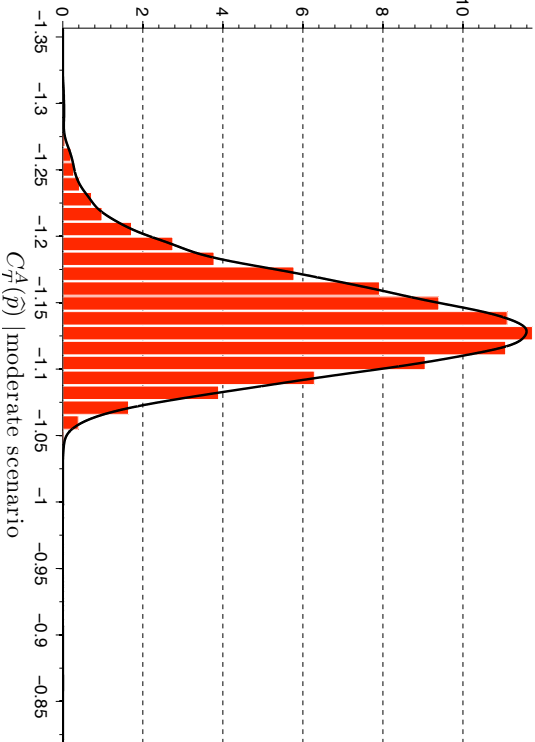
Figure 3.22: Overview of the statistics of the optimal liquidity costs $C_T^A(p)$, the three liquidity call scenarios and the corresponding unconditional probabilities, and the histograms of the conditional distributions for the universal bank (simulation trials = 250,000). In all three figures the optimal liquidity costs are expressed in units of initial capital.

Optimal liquidity costs		
Expected value	6.32%	
$VaR_{0.99}/\ell_0$	118.81%	
$ES_{0.99}/\ell_0$	135.57%	
Liquidity calls		
	$\alpha(\omega)/V_0(p)$	$\mathbb{P}\{\omega\}$
no call	0.00%	94.60%
moderate	44.98%	4.79%
extreme	48.98%	0.61%

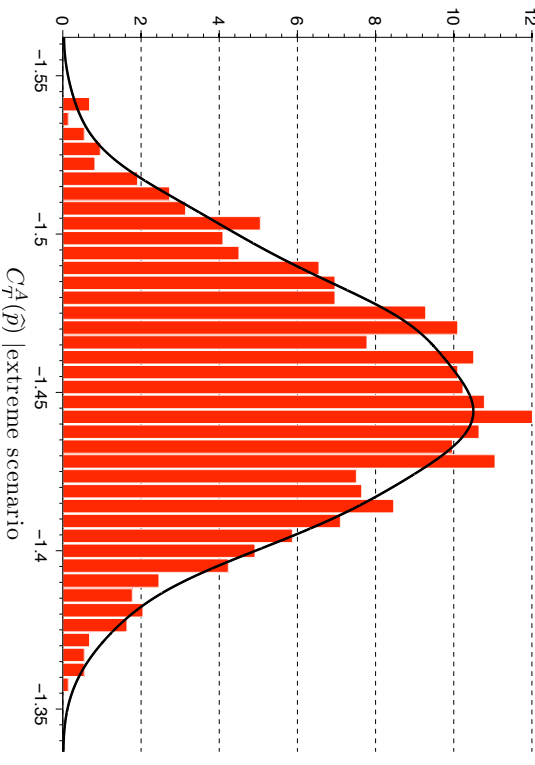
(a) Investment bank: Statistics



(b) Investment bank: Conditional optimal liquidity costs



(c) Investment bank: Cond. optimal liq. costs under moderate scenario



(d) Investment bank: Cond. optimal liq. costs under extreme scenario

Figure 3.23: Overview of the statistics of the optimal liquidity costs $C_T^A(p)$, the three liquidity call scenarios and the corresponding unconditional probabilities, and the histograms of the conditional distributions for the investment bank (simulation trials = 250,000). In all three figures the optimal liquidity costs are expressed in units of initial capital.

(see Theorem 2.26 on p. 38). The super-homogeneity effect can be explained by the convexity of the liquidity call function and the concavity of the proceed functions. The former deals with the fact that liquidity calls at least grow linearly with portfolio size and the latter refers to the fact that the more a bank has to sell in difficult times the less it gets per unit.

Since the only deviation from the setup in Chapter 2 is the inclusion of the non-positional operational losses (see Section 3.3 on p. 66), we should expect only minor distortions of the results. In Figure 3.24 on p. 101, Figure 3.25 on p. 102, and Figure 3.26 on p. 103 we plotted the scaling factor $\lambda \in (0, 2]$ against the VaR, ES, L-VaR, and L-ES divided by the initial capital for all three banks. We first notice that the VaR and ES as a fraction of the initial capital have the same shape for all banks. For λ values close to zero, the values are extremely large and then monotonically decrease over the whole range. The decrease at first is rapid, while the values level off but still decrease from λ values of around 0.3 and onwards. This behavior can perfectly be explained by the distorting effects of the operational losses. Since we assume that operational losses are independent of the portfolio size, it is clear that the VaR and ES as a fraction of the initial capital will be very large for small portfolios (near zero λ values), i.e., the initial capital is very small in comparison to the operational losses. Clearly, it would be more reasonable to have some form of scaling of operational losses but chose not to do it because there is no obvious way to do. The monotonically decreasing nature results from the fact that the operational losses have less and less of an impact due to the increasing capital. Clearly, without the distorting effects of operational risk, we would observe a horizontal line in all plots for the liquidity risk-free case, due to linearity.

The impact of bank size with liquidity risk follows a similar pattern for the universal bank and the investment bank. We see in Figure 3.25 on p. 102 and Figure 3.26 on p. 103, apart from the discussed initial op-risk effect, that L-VaR and L-ES increase for λ values greater than one. This result is in line with the general result of super-homogeneity. In our case this also means that increasing liquidity risk effect dominates the decreasing operational risk impact. The only difference between the two banks is that the super-homogeneity effect of liquidity risk is a great deal stronger for the investment bank than the universal bank. This difference is reasonable as the investment bank always has higher liquidity calls and relies more on fire selling, facing the concave proceed function, than the universal bank.

The retail bank shows some different pattern. In particular, in Figure 3.24 on p. 101 we see a bump at around $\lambda = 0.3$ in both the L-VaR and the L-ES plot. At these values the L-VaR jumps to 2000%, which is exactly amounts to a loss of the total assets over the initial capital:

$$\frac{V_0(p)}{\phi_{\text{equity}} V_0(p)} = \frac{1}{\phi_{\text{equity}}} = \frac{1}{0.05} = 20.$$

This means that the probability of the extreme scenario occurring must be larger or equal than the confidence level of the risk measure at these values, i.e., $\mathbb{P}\{\text{extreme scenario}\} \geq 1\%$, as we know that the retail bank defaults whenever the extreme scenario occurs

(see Figure 3.21 on p. 96). In other words, L-VaR for these λ values is sensitive to the default by illiquidity states and hence the 100%. The fact that the riskiness decreases for upscaling means that the increasing liquidity risk effect is dominated by the decreasing op-risk impact, which stands in contrast to the effect for the other two banks.

3.8 Conclusions

We have shown that even a simple but reasonable implementation of liquidity risk modeling leads to non-trivial results. In particular, we have shown that liquidity risk can lead to a significant deterioration of capital requirements and risk-adjusted performance for banks with safe funding but illiquid assets, exemplified by the retail bank, and banks with liquid assets but risky funding, exemplified by the investment bank. It is worth mentioning that the retail bank gets punished in our illustration mostly due to Type 2 liquidity risk. In addition, we have shown that the formal results of Theorem 2.26 on p. 38 are relevant, especially the super-homogeneity result of liquidity-adjusted risk measures. Overall bank size and the non-linear scaling effects of liquidity risk become very apparent for banks that have to rely on a large amount of fire selling as represented by the investment bank. Overall, our illustration confirms the common intuition that a bank's liquidity risk management must involve the careful balancing between the market liquidity risk of its assets and the funding risk of its liabilities. It may be helpful to think of the overall liquidity risk of a bank in terms of the occurrence frequency of liquidity calls, liquidity call severity, and the severity of the liquidity costs in case of liquidity calls. Overall, it is comforting that our liquidity risk formalism captures and brings forth the crucial dimensions of a bank's liquidity risk.

There appears to be a friction between what happened during the last banking crisis and some of our results in this illustration. In our analysis, while the investment bank was highly susceptible to liquidity calls and high liquidity costs, it did not suffer from default by illiquidity states due to their large trading portfolio. In reality, investment banks were the banks that defaulted. What are the differences between the characteristics of what we labeled an investment bank and the real world counterparts in question? While there could be numerous explanations for this discrepancy, a reasonable explanation is that in reality there was a markedly misjudgment of the market liquidity of the trading portfolio of the investment banks in times of crisis or at least in that particular situation. This might be explained by the fact that the real investment banks did not have the traditional "liquid" trading positions in their trading portfolio but also loan activities. In so far, the results of our analysis reflect the "ideal" idea of the liquidity risk characteristics of an investment bank: susceptible to funding risk but matched with a large marketability of its asset position.¹⁷ In that sense, it may be more reasonable to differentiate the stochastic modeling of the proceed functions and reflect the idea that the market liquidity risk is prone to jumps. Fortunately, this is no defect of

¹⁷In fact, this also described the special investment vehicles (SIV) that suffered from the same problems.

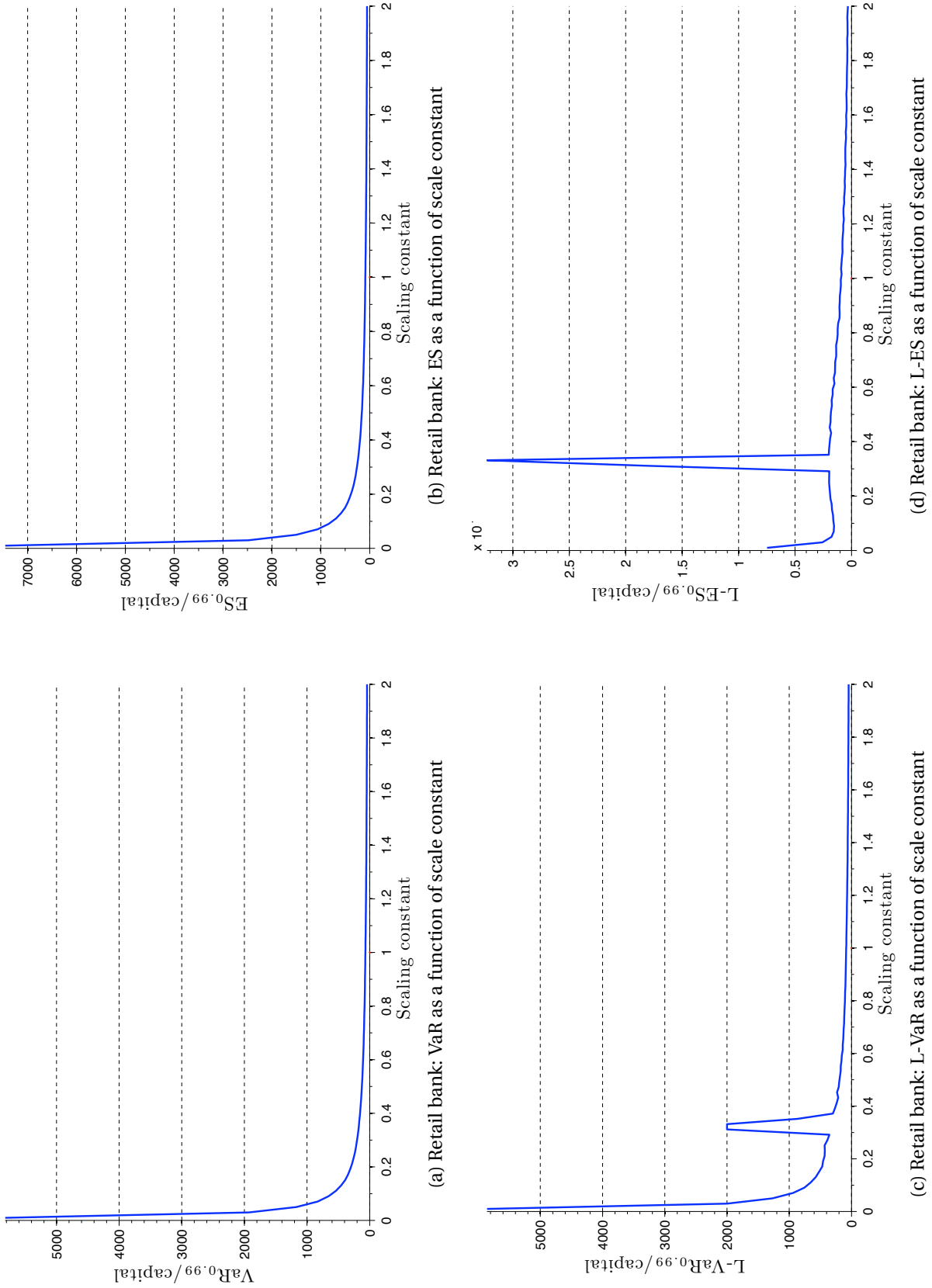


Figure 3.24: The conditional optimal liquidity cost for the universal bank.

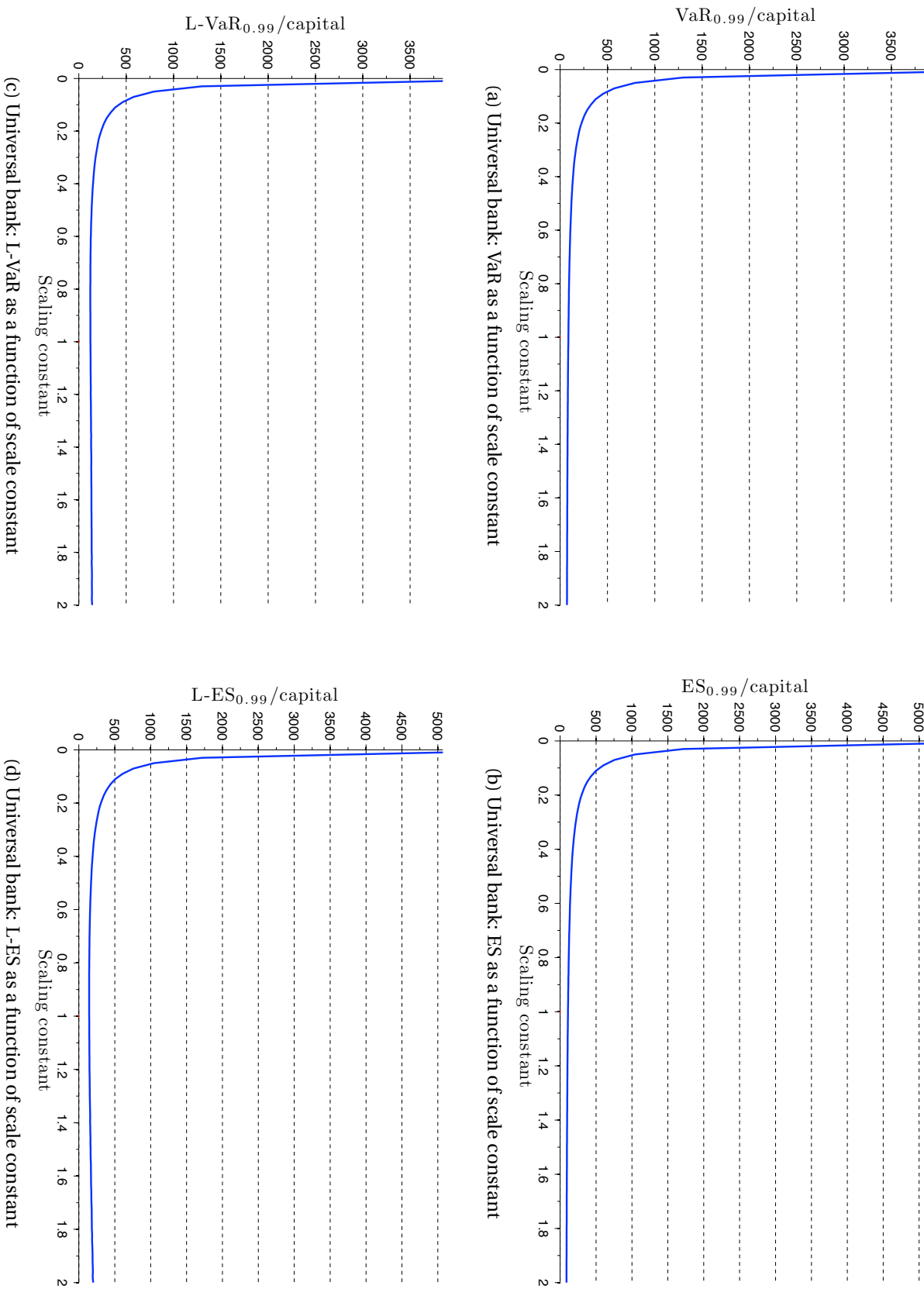
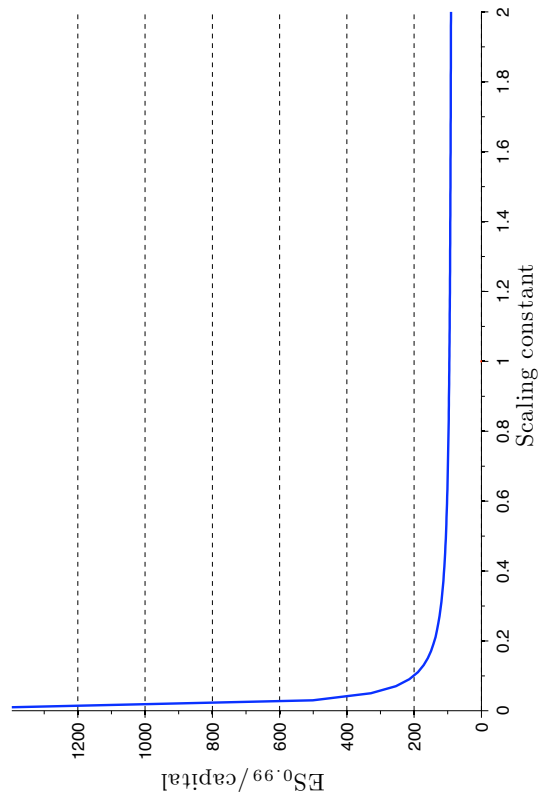
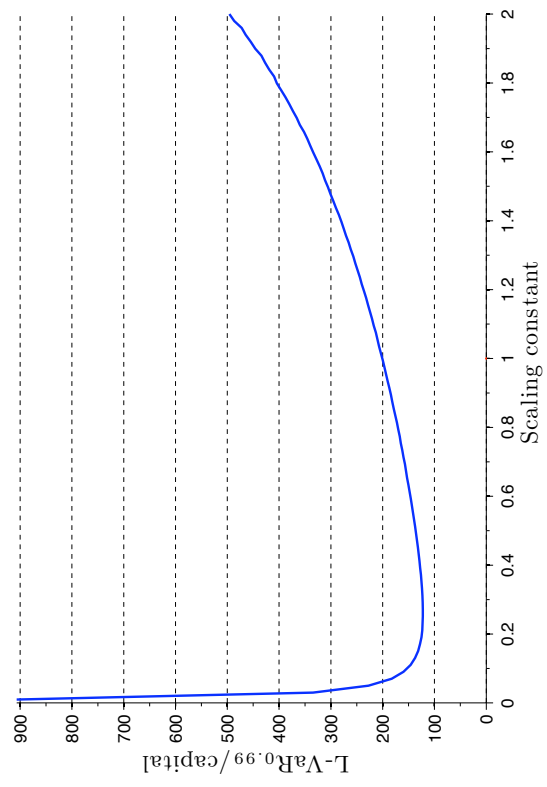


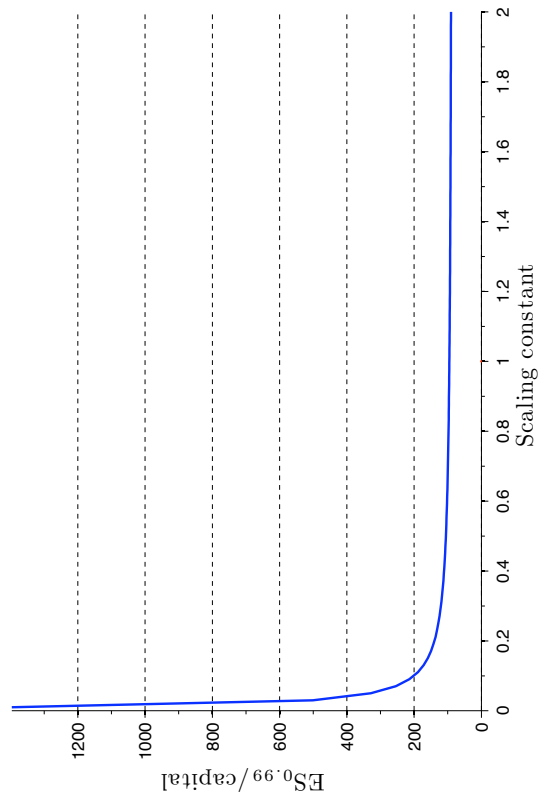
Figure 3.25: The conditional optimal liquidity cost for the universal bank.



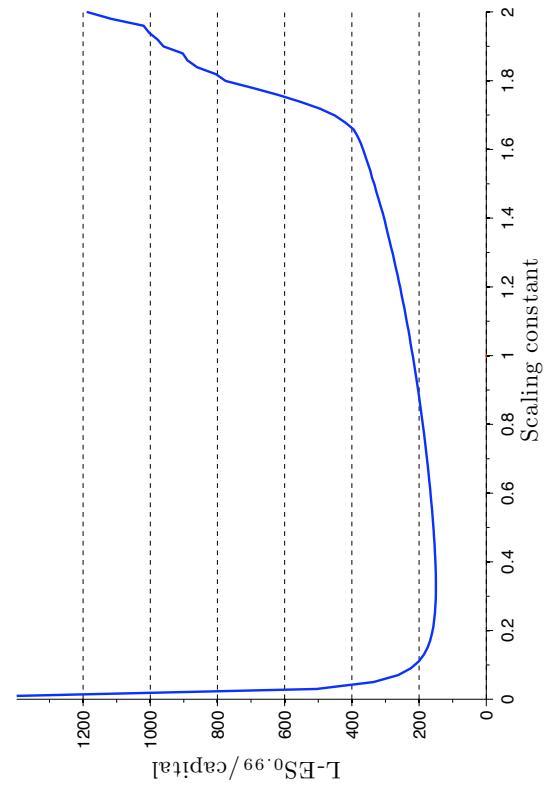
(a) Investment bank: VaR as a function of scale constant



(b) Investment bank: L-VaR as a function of scale constant



(c) Investment bank: ES as a function of scale constant



(d) Investment bank: L-ES as a function of scale constant

Figure 3.26: The conditional optimal liquidity cost for the universal bank.

our formalism but a question of model choice.

There is an interesting link between our modeling of a bank's funding liquidity risk and the novel securities known as contingent convertibles (CoCos). A CoCo is a bond that will automatically convert to core capital if a specific event occurs. For banks this usually involves a fall of the bank's Tier 1 capital below a specific level. It should be clear that in our illustration a bank having CoCos would directly reduce its funding liquidity risk due to a reduction in liquidity call sizes in stress scenarios.

While our illustration has been non-trivial, it is still overly stylized compared to the real-world EC models of banks. In case our liquidity risk formalism is deemed to be useful by practitioners, there is a need for better joint probability models for liquidity risk and solvency risk. However, there is often the sentiment that "it is difficult to quantify" liquidity risk and hence it should be analyzed predominately with the help of stress scenarios and not full probability models.¹⁸ A possible explanation for this line of reasoning is that people often believe that it only makes sense to use probability models in situations where relevant data is deemed to be redundant. Unfortunately, for liquidity risk modeling this is seldom the case. In contrast, we believe that a lack of "sufficient" statistical data should not automatically lead to abandoning the use of probability models. As soon as we fully embrace the Bayesian perspective after De Finetti and view probability theory "merely" as an useful mathematical tool to frame and represent our uncertainty about future events, we are not purely limited to the data-driven approaches anymore. Ideally, any form of "background knowledge" should be factored into the model choice and if the existing background information cannot single out a particular probability model in the model space, then there should be no hesitation to use ambiguity (second-order uncertainty) representations. While we believe that generalizing the uncertainty representation in this direction will be beneficial for a more systematic and quantitative liquidity risk management, we also realize that more research is needed before these ideas can be applied in practice in a systematic manner.

Appendix

A. Normal approximation for default count simulations

The de Moivre-Laplace Theorem (see, e.g., p. 186 in Feller (1968)) states that a binomial distribution $\mathcal{B}(n, p)$ can be approximated by $\mathcal{N}(np, np(1-p))$, if n is large and p is not too close to 0 or 1. We simulate the default counts of a category as follows:

Algorithm 3.2 Simulation of default counts using normal approximation.

1. Generate $Z_r \sim \mathcal{N}(\mu_r, \sigma^2)$, using the parameters in Table 3.2.

¹⁸Recall the earlier quote from BIS (2009) on p. 6 "Not all risks can be directly quantified. Material risks that are difficult to quantify in an economic capital framework (eg funding liquidity risk or reputational risk) should be captured in some form of compensating controls (sensitivity analysis, stress testing, scenario analysis or similar risk control processes)."

2. Compute vector of conditional default probabilities $q_r(z) = \Phi(Z_r)$, where Φ is the standard normal distribution function.
3. Generate vector of default counts $M_r | Z_r = z_r \sim \mathcal{B}(p_r^{\text{lb}}, \Phi(Z_r)) \approx \mathcal{N}(p_r^{\text{lb}}\Phi(Z_r), p_r^{\text{lb}}\Phi(Z_r)(1 - \Phi(Z_r)))$.

B. Sampling from a multivariate distribution using copulas

Here we briefly describe how to simulate realizations from a multivariate distribution, given we have a well-specified copula and marginal distributions, and we know how to generate realizations from the copula and the marginals. We can use the converse of Sklar's Theorem (see for example Theorem 5.3 in McNeil et al. (2005)) to sample from a joint distribution function as follows:

Algorithm 3.3 Simulation from multivariate distribution using copulas.

1. Generate d -dimensional random vector $U \sim C$, where $C : [0, 1]^d \rightarrow [0, 1]$ is the copula of the joint distribution function.
2. Return d -dimensional random vector $X := (F_1^{-1}(U_1), \dots, F_d^{-1}(U_d))$, where $F_1^{-1}, \dots, F_d^{-1}$ are the cumulative inverse distribution functions of the marginals.

The algorithm ensures that X has margins F_1, \dots, F_d and a multivariate distribution function

$$C(F_1(x_1), \dots, F_d(x_d)).$$

See Chapter 5 in McNeil et al. (2005) for a textbook treatment of copulas. For applying Algorithm 3.3, we need (a) to be able to generate samples from the copula and (b) have the cumulative inverse distribution function for all marginals available. The first requirement is rather straightforward as we are dealing with the easy to use Student's t copula, i.e., $C = C_{v,p}^t$. In most modern computer software packages, we can even rely on a built-in functions to do the job. For example, in Matlab we can use the function "copularnd" with the student-t option. The second requirement is not as straightforward as we cannot rely on built-in functions because we do not only have parametric marginals. There are several ways to approach the issue. We choose to use non-parametric kernel density estimation to estimate the "empirical" cumulative distribution function and then derive the inverse CDF. See also the next section regarding kernel density estimation. In Listing 3.1 we show our Matlab implementation.

```

1 function output = samplejoint(samples,rho,nu,num_trials)
2 % Synopsis
3 % -----
4 %     output = samplejoint(samples,rho,nu,num_trials)
5 % -----
6 % Description
7 % -----
8 %     samplejoint generates num_trials joint realizations
9 %     using a student-t copula, using KDE estimated
10 %     marginals of the samples realizations
11 % -----
12 % Inputs
13 % -----
14 %     (matrix) samples      : NxM matrix of marginal
15 %                           realizations
16 %     (matrix) rho          : MxM matrix of rank
17 %                           correlations
18 %     (scalar) nu           : degrees of freedom of
19 %                           student-t copula
20 %     (scalar) num_trials   : number of joint trials
21 % -----
22 % Outputs
23 % -----
24 %     (matrix) output       : (num_trials)xM joint
25 %                           realizations matrix
26 % -----

```

```

27 % Example
28 % -----
29 %     output = samplejoint(normrnd(0,1,10000,2),0.8,5,1000)
30 % -----
31 % Functions used
32 % -----
33 %     size, copularnd, ones, kde, sort, min, unique,
34 %     interp1
35 % -----
36 num_dim = size(samples,2);
37 U = copularnd('t',rho,nu,num_trials);
38 joint_samples = ones(num_trials,num_dim);
39 for i=1:num_dim
40     joint_samples(:,i) = kdeicdf(samples(:,i),U(:,i));
41 end
42 output = joint_samples;
43 end
44 % -----
45 function output = kdeicdf(samples,U) % Kernel estimate of
46 [-,~,x,cdf] = kde(samples);          % the inverse CDF
47 table = [sort(x)' min(1,cdf)];
48 [~,ind_unique,~] = unique(table(:,2),'first');
49 table = table(ind_unique,:); % delete duplicates
50 output = interp1(table(:,2),table(:,1),U);
51 end

```

Listing 3.1: Sampling joint realization from a matrix of marginal samples, using KDE estimates of the inverse CDFs and a student-t copula.

C. Kernel density estimation

In this chapter, we use *kernel density estimation* (KDE) to generate samples from the joint probability distribution. KDE is a non-parametric method of estimating the probability density function of a random variable. Given an independent and identical distributed sample of random variables $(x_1, \dots, x_n) \in \mathbb{R}^n$, $b > 0$, and a symmetric function K that integrates to one, the *kernel density estimator* is given by

$$f_h(x) := \frac{1}{nb} \sum_{i=1}^n K\left(\frac{x - x_i}{b}\right).$$

The function K is known as the *kernel function* and the parameter b is known as the *bandwidth*. There are several candidates for the kernel function, but very often the standard normal density function ϕ , i.e., $K(x) = \phi(x)$ is used for convenience. We also use it for our purposes. The choice of the bandwidth involves the trade-off between the bias and the variance of the estimator. We use a Matlab function (maintained by one of the authors) based on the ideas presented in Botev et al. (2010). It involves an automatic data-driven bandwidth selection that is generally better than the common rule-of-thumb approach, based on the assumption that the samples are normally distributed. We refer the interested readers to the article for more information.

Another important use of the KDE is the computation of the Euler allocation for VaR. We used this result in the simulation example of Chapter 2. Recall the definition of the VaR risk contribution:

$$\text{VaR}_\beta(p_i | p) = -E[p_i X_i | X(p) = -\text{VaR}_\beta(X(p))]. \quad (3.14)$$

As pointed out in Tasche (2008), we cannot simply substitute the “empirical” sample data from the simulation in Equation 3.14 to compute the risk contribution, because the conditioning event does not have a positive probability. However, we can use the KDE to our advantage. From Tasche (2008, 2000) we get the Nadaraya Watson kernel estimator of Equation 3.14:

$$\text{VaR}_\beta(p_i | p) \approx -\frac{\sum_{k=1}^n x_{i,k} K\left(\frac{-\text{VaR}_\beta(\hat{X} + b\xi) - x_k}{b}\right)}{\sum_{k=1}^n K\left(\frac{-\text{VaR}_\beta(\hat{X} + b\xi) - x_k}{b}\right)}. \quad (3.15)$$

D. Optimal liquidation algorithm

With the help of Lemma 2.10 it is possible to find analytical solutions for the optimal liquidity costs for our illustration per scenario. However, it can become cumbersome to account for position upper bounds and implement it in code when there are many different assets involved, possibly with different functional forms of proceed functions. Fortunately, we can avoid such problems by using a simple, robust, and efficient numerical *grid-search algorithm* based on Lemma 2.10. For the latter we simply create a *look-up table* as shown in Table 3.5 and find for a given α the corresponding optimal liquidity costs. There is no harm in using the numerical search algorithm because

we do not have to worry about local optima because we are dealing with a convex optimization problem.

D.1 Some analytical solutions

Before we turn towards the algorithm, we show why it can become cumbersome to derive and code the analytical solutions directly.

Example 3.4 Linear proceeds. Suppose we have a portfolio $p \in \mathcal{P}$ and all N proceed functions are linear functions:

$$G_i(x_i) = \beta_i V_i x_i \text{ with } 0 \leq \beta_i \leq 1 \text{ for } i = 1, \dots, N.$$

Recall that the optimal liquidity costs are given by:

$$C^\alpha(p) = \min\{C(x) \mid G(x) = \alpha \text{ and } 0 \leq x \leq p\}.$$

For the solution in the linear situation, we do not even need the help of Lemma 2.10, but can directly solve it. Consider the indices are assigned so that $i > j \implies \beta_i \geq \beta_j$. In other words, we now have that $\beta_0 \geq \beta_1 \geq \dots, \beta_N$. The optimal liquidation strategy is then given by

$$x_{\text{lin}}^*(p, \alpha) = \begin{cases} (\alpha, \vec{0}) & \text{for } \alpha \leq p_0 \\ (p_0, \frac{\alpha - p_0}{V_1 \beta_1}, \vec{0}) & \text{for } p_0 < \alpha \leq G(p_0, p_1, \vec{0}) \\ (p_0, p_1, \frac{\alpha - G(p_0, p_1, \vec{0})}{V_2 \beta_2}, \vec{0}) & \text{for } G(p_0, p_1, \vec{0}) < \alpha \leq G(p_0, p_1, p_2, \vec{0}) \\ \vdots & \vdots \\ (p_0, \dots, p_{N-1}, \frac{\alpha - G(p_0, \dots, p_{N-1}, 0)}{V_N \beta_N}) & \text{for } G(p_0, \dots, p_{N-1}, 0) < \alpha \leq G(p) \\ \emptyset & \text{for } \alpha > G(p), \end{cases}$$

and the optimal liquidity costs by $C^\alpha(p) = V(x_{\text{lin}}^*(p, \alpha)) - \alpha$. For a numerical example, consider we have that $N = 2$ and a portfolio given by $p = (p_0, p_1, p_2) = (0, 10, 10)$. Furthermore, we have that $V_1 = 8$, $V_2 = 10$, $V(p) = p_1 V_1 + p_2 V_2 = 180$, $\alpha = 100$, $\beta_1 = 0.8$, and $\beta_2 = 0.6$. The optimal liquidation strategy is

$$x_{\text{lin}}^*(p, 100) = (0, 10, \frac{10}{3})$$

and the optimal liquidity costs are

$$C(x_{\text{lin}}^*(p, 100)) = C^\alpha(p) \approx 113.33 - 100 = 13.33.$$

Example 3.5 Exponential proceeds. Now suppose we have a portfolio $p \in \mathcal{P}$ and all N proceed functions are exponential proceed functions:

$$G_i(x_i) = \frac{V_i}{\theta_i} (1 - e^{-\theta_i x_i}) \text{ with } \theta_i \geq 0 \text{ for } i = 1, \dots, N.$$

Using Lemma 2.10, it is straightforward to show that the optimal liquidation strategy as a function of μ is given by

$$x^*(\mu) = (x_1^*, \dots, x_N^*) = (-\frac{\ln \mu}{\theta_1}, \dots, -\frac{\ln \mu}{\theta_N}),$$

if we do not take into account position upper bounds. Furthermore, we have that

$$\mu_i(x_i) = \frac{G'_i(x_i)}{V_i} = e^{-\theta_i x_i},$$

$$\alpha(\mu) = \sum_{i=1}^N \frac{V_i}{\theta_i} (1 - \mu),$$

and

$$\mu(\alpha) = 1 - \left(\sum_{i=1}^N \frac{V_i}{\theta_i} \right)^{-1} \alpha.$$

Now the optimal liquidation strategy as a function of α of the i th position is given by

$$\tilde{x}_{i,\text{exp}}^*(\alpha) = -\frac{\ln(1 - (\sum_{i=1}^N \frac{V_i}{\theta_i})^{-1} \alpha)}{\theta_i},$$

and the whole optimal liquidation strategy by $\tilde{x}_{\text{exp}}^*(\alpha) = (\tilde{x}_{1,\text{exp}}^*(\alpha), \dots, \tilde{x}_{N,\text{exp}}^*(\alpha))$. The optimal liquidity costs as a function of α for a given p are then given by

$$\begin{aligned} C^\alpha(p) &= C(\tilde{x}_{\text{exp}}^*(\alpha)) = V(\tilde{x}_{\text{exp}}^*(\alpha)) - G(\tilde{x}_{\text{exp}}^*(\alpha)) \\ &= V(\tilde{x}_{\text{exp}}^*(\alpha)) - \alpha. \end{aligned}$$

However, we need to take into account the position upper bounds of the portfolio. In order to achieve this, consider the following functions for given $p \in \mathcal{P}$:

$$\tilde{\mu}_i := \mu_i(p_i) \text{ for } i = 1, \dots, N.$$

Now consider we reassign the indices so that $i > j \implies \tilde{\mu}_i \geq \tilde{\mu}_j$. For every $\tilde{\mu}_i$ there is a corresponding $\tilde{\alpha}_i$ given by

$$\tilde{\alpha}_i := \sum_{j=i}^N \frac{V_j}{\theta_j} (1 - \tilde{\mu}_i) + \sum_{k=1}^i G_{k-1}(p_{k-1}).$$

Note that we have that $\tilde{\alpha}_1 \leq \tilde{\alpha}_2 \leq \dots \leq \tilde{\alpha}_N$. The optimal liquidation strategy for the constrained case is then given by

$$x^*(\alpha) = \begin{cases} (\alpha, \vec{0}) & \text{for } \alpha \leq p_0 \\ (p_0, \tilde{x}_{1,\text{exp}}^*(\alpha), \dots, \tilde{x}_{N,\text{exp}}^*(\alpha)) & \text{for } p_0 < \alpha \leq \tilde{\alpha}_1 \\ (p_0, p_1, \tilde{x}_{2,\text{exp}}^*(\alpha), \dots, \tilde{x}_{N,\text{exp}}^*(\alpha)) & \text{for } \tilde{\alpha}_1 < \alpha \leq \tilde{\alpha}_2 \\ \vdots & \vdots \\ (p_0, \dots, p_{N-1}, \tilde{x}_{N,\text{exp}}^*(\alpha)) & \text{for } \tilde{\alpha}_{N-1} < \alpha \leq \tilde{\alpha}_N \end{cases}.$$

For a numerical example, consider we have that $N = 2$ and a portfolio given by $p = (p_0, p_1, p_2) = (0, 10, 10)$. In addition, we have that $V_1 = 8$, $V_2 = 10$, $V(p) = p_1 V_1 + p_2 V_2 = 180$, $\alpha = 100$, $\theta_1 = 0.02$, and $\theta_2 = 0.08$ $C^{100}(p) = 11.05$ $x^*(100) = (0, 10, 3.1)$.

$$\mu_1(p_1) = e^{-0.02 \cdot 10} \approx 0.82$$

$$\mu_2(p_2) = e^{-0.08 \cdot 10} \approx 0.45$$

μ	x_1^*	...	x_N^*	$G(x^*) = \alpha$	$C(x^*) = C^\alpha(p)$
1					
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
0					

Table 3.5: Look-up table behind the numerical algorithm for finding the optimal liquidity costs.

$$\tilde{\alpha}_1 = \sum \frac{V_i}{\theta_i} (1 - \mu_1(p_1)) \approx 525(1 - 0.82) = 94.5$$

Hence, we generate 94.5 cash using the unconstrained optimal strategy, but then asset 1 is exhausted and we need to generate the rest $100 - 94.5 = 5.5$ solely from the remaining asset 2 position.

$$\begin{aligned} C^{100}(p) &= V(x^*(\tilde{\alpha}_1)) + V\left(-\frac{\ln(1 - (\frac{V_2}{\theta_2})^{-1}(\alpha - \tilde{\alpha}_1))}{\theta_2}\right) - \alpha \\ &\approx 8 \cdot p_1 + 10 \cdot \frac{\theta_1 p_1}{\theta_2} + 10\left(-\frac{\ln(1 - 5.5/125)}{0.08}\right) - 100 \\ &\approx 80 + 25 + 5.62 - 100 \\ &\approx 10.43 \end{aligned}$$

While it is possible to find analytical solutions and turn them into algorithm, we provide a simpler solution that can readily adjusted for different forms of proceed functions.

D.2 Grid-search algorithm

The algorithm involves the following five steps:

1. Set up a grid of μ values between 0 and 1. The grid does not have to be evenly spaced.
2. Evaluate the optimal liquidation strategy per asset position at each grid point, taking into account position upper bounds.
3. Compute for each grid point the corresponding α and optimal liquidity costs, using the vector of optimal liquidation positions.
4. Collecting these values and create a look-up table.
5. Search for the optimal liquidity costs for a given α in the look-up table, using some form of interpolation method.

The resulting look-up table is shown in Table 3.5. We illustrate the workings of the algorithm in the following example.

Example 3.6 Algorithm with exponential proceeds. Consider the same data as in Example 3.5. Recall that

$$\mu_i(x_i) = \frac{G'_i(x_i)}{V_i} = e^{-\theta_i x_i}.$$

It is straightforward to solve the above equation for x_i :

$$x_i = \frac{1}{-\theta_i} \ln(\mu).$$

Now we can create a table that gives us for any μ the optimal liquidation strategies and the corresponding α and optimal liquidity costs, taking into account the position upper bounds:

μ	$x_1^* = \min(x_1, p_1)$	$x_2^* = \min(x_2, p_2)$	$G(x^*) = \alpha$	$C^\alpha(p)$
1.0	0	0	0	0
0.8	10	2.79	97.51	10.39
0.6	10	6.39	122.51	21.35
0.4	10	10	141.34	38.66
0.2	10	10	141.34	38.66
0.0	10	10	141.34	38.66

Finally, we can leave out the rows beyond the row with $\mu = 0.4$ because the portfolio is illiquid at that point.

μ	x_1^*	x_2^*	α	$C^\alpha(p)$
1.0	0	0	0	0
0.8	10	2.79	97.51	10.39
0.6	10	6.39	122.51	21.35
0.4	10	10	141.34	38.66

Example 3.7 Algorithm - exponential and linear proceeds. Now consider we add 10 units of an asset that has a linear proceed function to the portfolio:

$$G_1(x_1) = \theta_1 V_1 x_1,$$

with $\theta_1 = 0.9$ and $V_1 = 10$. Because we have that

$$\mu_1(x_1) = \theta_1,$$

we have that

$$x_1^* = p_1 \mathbf{1}_{\{\theta_1 = \mu\}}.$$

The table with a finer-grained μ grid is given in Table 3.6. For using the table as a table look-up, it is useful to use an interpolation method, e.g., linear interpolation. Apart from its efficiency and simplicity, we recommend the linear method in our case because it avoids some undesirable results other methods such as splines have for assets with linear proceeds. The problem involves large jumps in α in the table when linear assets are present. Note the jump in α in the above example from just before the use of the first asset to using the it: 26.25 to 124.50. The linear method correctly assigns the same μ for α s between the two points.

μ	$x_1^* = p_1 \mathbf{1}_{\{\theta_1 = \mu\}}$	$x_2^* = -\frac{1}{0.02} \ln(\mu)$	$x_3^* = -\frac{1}{0.08} \ln(\mu)$	α	$C^\alpha(p)$
1.00	0.00	0.00	0.00	0.00	0.00
0.95	0.00	2.56	0.64	26.25	0.68
0.90	10.00	5.27	1.32	124.50	10.81
0.85	10.00	8.13	2.03	150.75	14.57
0.80	10.00	10.00	2.79	169.51	18.39
0.75	10.00	10.00	3.60	175.76	20.20
0.70	10.00	10.00	4.46	182.01	22.58
0.65	10.00	10.00	5.38	188.26	25.59
0.60	10.00	10.00	6.39	194.51	29.35
0.55	10.00	10.00	7.47	200.76	33.97
0.50	10.00	10.00	8.66	207.01	39.64
0.45	10.00	10.00	9.98	213.26	46.56
0.40	10.00	10.00	10.00	213.34	46.66

Table 3.6: Look-up table for Example 3.7.

D.3 Matlab implementation

Here we show a Matlab implementation of the above algorithm. The code allows for an arbitrary mix of assets with exponential proceed functions and assets with linear proceeds.

```

52 function out = mugridtable(A,B,grid_step)
53 % -----
54 % Synopsis
55 % -----
56 %     out = mugridtable(A,B,grid_step)
57 % -----
58 % Description
59 % -----
60 %     mugridtable computes look-up table that can used for
61 %     the computation of the optimal liquidity costs,
62 %     value, and strategy for any number of combination of
63 %     linear and exponential proceed functions
64 % -----
65 % Inputs
66 % -----
67 %     (matrix)      A : 3xN matrix
68 %     (matrix)      A : 3xM matrix
69 %     (scalar) grid_step : determines the granularity of
70 %                         the mu grid (0,1)
71 % -----
72 % Outputs
73 % -----
74 %     (matrix) out : Lx(N+M+3) matrix representing the
75 %                   look-up table, where L =
76 %                   length(mu_grid);
77 %                   out(:,1)      mu grid
78 %                   out(:,2:N+1)  optimal strategy
79 %                                 for assets with
80 %                                 linear proceeds
81 %                                 per mu
82 %                   out(:,N+1:end-2) optimal strategy
83 %                                 for assets with
84 %                                 linear proceeds
85 %                                 per mu
86 %                   out(:,end-1)  liquidity call
87 %                                 per mu
88 %                   out(:,end)    optimal liquidity
89 %                                 cost per mu
90 % -----
91 % Example

```

```

92 % -----
93 %     mugridtable([10;10;0.9],[10;10;0.08],0.5)
94 %     ans =
95 %           1.0000         0         0         0         0
96 %           0.5000    10.0000     8.6643    152.5000    34.1434
97 %           0     10.0000    10.0000    158.8339    41.1661
98 % -----
99 % Functions used
100 % -----
101 %     inputParser, all, isfinite, size, isscalar, length,
102 %     ones, linproceeds, log, expproceeds
103 % -----
104 % References
105 % -----
106 %     [1] Loebnitz, K. and Roorda, B. (2011). Liquidity
107 %         Risk meets Economic Capital and RAROC.
108 %         http://ssrn.com/abstract=1853233.
109 % -----
110 % Author
111 % -----
112 %     Kolja Loebnitz <k.loebnitz@utwente.nl>
113 % -----
114 % License
115 % -----
116 %     The program is free for non-commercial academic use.
117 %     Please contact the author if you are interested in
118 %     using the software for commercial purposes. The
119 %     software must not modified or re-distributed without
120 %     prior permission of the authors.
121 % -----
122 % Changes
123 % -----
124 %     2011/09/19
125 p = inputParser;
126 p.addRequired('A',@(x) all(isfinite(x(:))) &&...
127               size(x,1)==3 && all(x(:)>0));
128 p.addRequired('B',@(x) all(isfinite(x(:))) &&...
129               size(x,1)==3 && all(x(:)>0));
130 p.addRequired('grid_step',@(x) isscalar(x) &&...
131               x>0 && x<1);

```

```

132 p.parse(A,B,grid_step);
133 A = p.Results.A;
134 B = p.Results.B;
135 grid_step = p.Results.grid_step;
136 %——Collect and declare parameters/variables——
137 posLin = A(1,:); pricesLin = A(2,:); thetaLin = A(3,:);
138 posExp = B(1,:); pricesExp = B(2,:); thetaExp = B(3,:);
139 num_assets_lin = length(posLin);
140 num_assets_exp = length(posExp);
141 mu_grid = 1:-grid_step:0; grid_size = length(mu_grid);
142 %——Compute optimal str. & proceeds (lin. proceeds)——
143 tmp_1 = mu_grid(:) ones(1,num_assets_lin); % Compute opt.
144 tmp_2 = thetaLin(ones(1,grid_size),:); % liq. str.(x)
145 strat_lin = posLin(ones(1,grid_size),:); % for a given
146 strat_lin(tmp_1>tmp_2) = 0; % mu grid
147 tmp_3 = ones(grid_size,1) pricesLin; % Compute
148 [proceeds_lin,~] = linproceeds(tmp_3,tmp_2,... % proceeds
149 strat_lin);
150 %——Compute optimal str. & proceeds (exp. proceeds)——
151 strat_exp_u = ones(grid_size,num_assets_exp);
152 for i=1:num_assets_exp % compute
153 strat_exp_u(:,i) = (-1/thetaExp(i)) ... % unconstr.

```

```

154 log(mu_grid); % x for
155 end % mu grid
156 strat_exp_c = min(strat_exp_u,... % adj. for
157 posExp(ones(1,grid_size),:)); % pos. constr.
158 tmp_4 = ones(grid_size,1) pricesExp;
159 tmp_5 = ones(grid_size,1) thetaExp;
160 [proceeds_exp,~] = exproceeds(tmp_4,tmp_5,strat_exp_c);
161 %——Combine results and create look-up table——
162 total_proceeds = proceeds_lin + proceeds_exp;
163 opt_liq_costs = strat_lin pricesLin' +...
164 strat_exp_c pricesExp'...
165 - total_proceeds;
166 table = [mu_grid' strat_lin strat_exp_c...
167 total_proceeds opt_liq_costs];
168 liquidation_value = linproceeds(pricesLin,thetaLin,...
169 posLin) + exproceeds(pricesExp,...
170 thetaExp,posExp);
171 [row_default,~] = find(total_proceeds >=...
172 liquidation_value,1);
173 %——Function output——
174 out = table(1:row_default,:);
175 end

```

Listing 3.2: Matlab implementation of the “grid-search” liquidation algorithm. Computation of the look-up table.

```

176 function [costs, val, mu, ...
177           lam, str] = optliquidity(cash, posLin, ...
178                                 pricesLin, thetaLin, ...
179                                 posExp, pricesExp, ...
180                                 thetaExp, alpha, ...
181                                 grid_step)
182 % -----
183 % Synopsis
184 % -----
185 % [costs, val, mu, ...
186 %   lam, str] = optliquidity(cash, posLin, pricesLin, ...
187 %                           thetaLin, posExp, pricesExp, ...
188 %                           thetaExp, alpha, grid_step)
189 % -----
190 % Description
191 % -----
192 %   optliquidity computes the opt. liq. costs, liq.-adj.
193 %   value, optimal liq. strategy, and the marginal costs
194 %   of an asset portfolio, given a liquidity call of
195 %   size alpha.
196 % -----
197 % Inputs
198 % -----
199 %   (scalar) cash      : cash position (non-negative)
200 %   (vector) posLin    : 1xN vector repr. the position
201 %                       for assets with linear proc.
202 %   (vector) pricesLin : 1xN vector representing asset
203 %                       prices
204 %   (vector) thetaLin  : 1xN vector representing the
205 %                       friction para.
206 %   (vector) posExp    : 1xN vector repr. the position
207 %                       for assets with exp. proc.
208 %   (vector) pricesExp : 1xN vector representing asset
209 %                       prices
210 %   (vector) thetaExp  : 1xN vector representing the
211 %                       friction para.
212 %   (scalar) alpha     : Liquidity call size (non-neg.
213 %                       real number)
214 %   (scalar) grid_step : level of granularity of grid
215 %                       (0,1)

```

```

216 % -----
217 % Outputs
218 % -----
219 %   (scalar) costs    : optimal liquidity costs
220 %   (scalar) val      : liquidity-adjusted portfolio
221 %                       value
222 %   (scalar) mu       : mu given portfolio at alpha [0,1]
223 %   (scalar) lam      : lambda given portfolio at alpha
224 %   (vector) str      : 1xN vector of optimal liquidation
225 %                       strategy
226 % -----
227 % Example
228 % -----
229 %   [costs val mu lam str] = optliquidity(0,10,10, ...
230 %                                       0.95,10,10, ...
231 %                                       0.08,150,0.001)
232 %
233 %           costs = 22.4773
234 %           val = 177.5227
235 %           mu = 0.5600
236 %           lam = 0.7857
237 %           str = 0   10.0000   7.2477
238 % -----
239 % Functions used
240 % -----
241 %   inputParser, isscalar, isfinite, isvector, all,
242 %   length, ones, linproceeds, exproceeds, min, zeros,
243 %   mugridtable, interp1q, find
244 % -----
245 % References
246 % -----
247 %   [1] Loebnitz, K. and Roorda, B. (2011). Liquidity
248 %       Risk meets Economic Capital and RAROC.
249 %       http://ssrn.com/abstract=1853233.
250 % -----
251 % Author
252 % -----
253 %   Kolja Loebnitz <k.loebnitz@utwente.nl>
254 % -----
255 % License

```

```

256 % The program is free for non-commercial academic use.
257 % Please contact the author if you are interested in
258 % using the software for commercial purposes. The
259 % software must not be modified or re-distributed without
260 % prior permission of the authors.
261 % -----
262 % Changes
263 % -----
264 % 2011/09/19
265 p = inputParser;
266 p.addRequired('cash', @(x) isscalar(x) &&...
267             isfinite(x) && x>=0);
268 p.addRequired('posLin', @(x) all(isfinite(x(:))) &&...
269             isvector(x) && all(x>0));
270 p.addRequired('pricesLin', @(x) all(isfinite(x(:))) &&...
271             isvector(x) && all(x>0));
272 p.addRequired('thetaLin', @(x) all(isfinite(x(:))) &&...
273             isvector(x) && all(x>0) && all(x<=1));
274 p.addRequired('posExp', @(x) all(isfinite(x(:))) &&...
275             isvector(x) && all(x>0));
276 p.addRequired('pricesExp', @(x) all(isfinite(x(:))) &&...
277             isvector(x) && all(x>0));
278 p.addRequired('thetaExp', @(x) all(isfinite(x(:))) &&...
279             isvector(x) && all(x>0));
280 p.addRequired('alpha', @(x) isscalar(x) &&...
281             isfinite(x) && x>=0);
282 p.addRequired('grid_step', @(x) isscalar(x) &&...
283             isfinite(x) && x>0 && x<1);
284 p.parse(cash, posLin, pricesLin, thetaLin, posExp, ...
285         pricesExp, thetaExp, alpha, grid_step);
286 cash = p.Results.cash; posLin = p.Results.posLin;
287 pricesLin = p.Results.pricesLin;
288 thetaLin = p.Results.thetaLin; posExp = p.Results.posExp;
289 pricesExp = p.Results.pricesExp;
290 thetaExp = p.Results.thetaExp; alpha = p.Results.alpha;
291 grid_step = p.Results.grid_step;

```

```

292 %-----Declare basic variables-----
293 num_assets_lin = length(posLin);
294 num_assets_exp = length(posExp);
295 num_assets = num_assets_lin + num_assets_exp;
296 no_friction_1 = ones(1, num_assets_lin);
297 no_friction_2 = ones(1, num_assets_exp);
298 MtM_value = linproceeds(pricesLin, no_friction_1, ...
299                        posLin) + ...
300                        linproceeds(pricesExp, no_friction_2, posExp);
301 liquidation_value = linproceeds(pricesLin, thetaLin, ...
302                                posLin) + ...
303                                expproceeds(pricesExp, thetaExp, ...
304                                             posExp);
305 %-----Function output-----
306 if alpha <= cash
307     costs = 0; val = MtM_value + cash;
308     mu = 1; lam = 0;
309     str = [min(cash, alpha) zeros(1, num_assets)];
310 elseif (alpha - cash) <= liquidation_value
311     A = [posLin; pricesLin; thetaLin];
312     B = [posExp; pricesExp; thetaExp];
313     adjustedTable = mugridtable(A, B, grid_step);
314     costs = interp1q(adjustedTable(:, end-1), ...
315                    adjustedTable(:, end), alpha-cash);
316     val = MtM_value + cash - costs;
317     mu = interp1q(adjustedTable(:, end-1), ...
318                adjustedTable(:, 1), alpha-cash);
319     lambda = @(x) (1-x)./x; lam = lambda(mu);
320     opt_str = adjustedTable(:, 2:end-2);
321     mu_grid = 1:-grid_step:0;
322     str = [cash opt_str(find(mu_grid'<=mu, 1), :)];
323 else
324     costs = NaN; val = 0; mu = NaN; lam = NaN;
325     str = NaN ones(1, num_assets);
326 end
327 end

```

Listing 3.3: Computation of optimal liquidity costs, the liquidity-adjusted value, the optimal liquidation strategy, and the marginal costs.

References

- BIS (2009). Range of practices and issues in economic capital frameworks. Bank of International Settlements. 18
- Botev, Z. I., Grotowski, J. F., and Kroese, D. P. (2010). Kernel density estimation via diffusion. *Annals of Statistics*, 38:2916 – 2957. 3.8
- Demarta, S. and McNeil, A. (2005). The t copula and related copulas. *International Statistical Review*, 73:111–130. 3.6.2
- Feller, W. (1968). *An Introduction to Probability Theory and Its Applications (Volume 1)*. Wiley. 3.8
- González, F., Cossin, D., Huang, Z., and Aunon-Nerin, D. (2003). A framework for collateral risk control determination. <http://ssrn.com/abstract=376233>. Working paper ECB, January. 3.4.1
- Hull, J. C. (2010). *Risk Management and Financial Institutions*. Pearson Education. 15
- Matz, L. and Neu, P. (2007). *Liquidity Risk Measurement and Management - A practitioner's guide to global best practices*. Wiley. 3.5
- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press. 3.4.2, 3.1, 3.4.2, 3.4.2, 3.2, 3.6.2, 13, 3.8, 3.8
- Rosenberg, J. V. and Schuermann, T. (2006). A general approach to integrated risk management with skewed, fat-tailed risks. *Journal of Financial Economics*, 79:569–614. 3.4.4, 3.3, 3.6.2
- Tasche, D. (2000). Risk contributions and performance measurement. www-m4.ma.tum.de/pers/tasche/riskcon.pdf. Working paper, February. 3.8
- Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the euler principle. In Resti, A., editor, *Pillar II in the New Basel Accord: The Challenge of Economic Capital*, pages 423–453. Risk Books. 3.8
- Vasicek, O. (1977). An equilibrium characterisation of the term structure. *Journal of Financial Economics*, 5:177–188. 3.4.1

4

Extensions of the framework

In this chapter, we discuss briefly several extensions of the basic framework presented in Chapter 2. In particular, we consider more complicated proceed functions that allow for cross-effects and permanent price impacts, an alternative capital allocation scheme, and the introduction of dynamics.

4.1 Asset cross-effects and permanent price impact

In the framework we presented in Chapter 2 we do not take into account asset cross-effects and permanent price impacts for expository purposes. Here we briefly discuss extending our formalism in these directions. Recall that we assumed that asset proceed functions $G_i \in \mathcal{G}$ are mappings of the form $G_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and that portfolio proceeds are simply the sum of the asset proceed functions:

$$G(p) = \sum_{i=0}^N G_i(p_i).$$

Asset cross-effects deal with the possibility that liquidating one asset might influence the proceeds of liquidating other assets. This influence can be positive or negative. Cross-effects describe dependent secondary asset markets and can allow us to formalize forms of “spill-over” effects (cf., for instance, Schoenborn (2008)). Formally, we can simply remove the assumption that the portfolio proceeds are the sum of the individual asset proceed functions. In particular, let us consider only negative cross-effects and that the portfolio proceeds adjusted for these effects have an additive form:

$$\tilde{G}(x) := G(x) - \text{Cross}(x),$$

where G is a portfolio proceed function and $\text{Cross} : \mathcal{P} \rightarrow \mathbb{R}_+$ so that for all i

$$\text{Cross}(0, \dots, p_i, \dots, 0) = 0,$$

and for $x, y \in \mathcal{P}$

$$x \geq y \implies \text{Cross}(x) \geq \text{Cross}(y).$$

The idea of asset cross-effects is best illustrated by a simple example.

Example 4.1 Simple incorporations of asset cross-effect. Suppose we have a portfolio proceed function given by $G(x) = x_0 + \sum_{i=1}^N V_i / \theta_i (1 - e^{-\theta_i x_i})$. Clearly, we have that $G_i \in \mathcal{G}$ for all i . Consider the following formalization of asset cross-effects:

$$\begin{aligned} \tilde{G}(x) &= G(x) - \text{Cross}_i(x), \\ &= G(x) - \sum_{i=1}^N G_i(x_i)(1 - e^{-\beta_i \bar{x}_i}), \end{aligned}$$

where $\bar{x}_i := \sum_{j \neq i} x_j$ and $\beta_i \geq 0$ for all i and $\beta_0 = 0$. Let us consider a simple numerical example. Assume that $N = 2$ and suppose we have the following parameter values: $V_1 = 10, V_2 = 10, \theta_1 = 0.04, \theta_2 = 0.08, \beta_1 = 0.01$, and $\beta_2 = 0.05$. Plots of the total proceeds with and without asset cross-effects as a function of the two assets can be found in Figure 4.1. We assume that the cash position is zero. ■

It should be clear that introducing the asset cross-effects will have an impact on Lemma 2.10. We think that asset cross-effects may be a valuable element in an analysis of systemic risk involving multiple banks and liquidity risk.

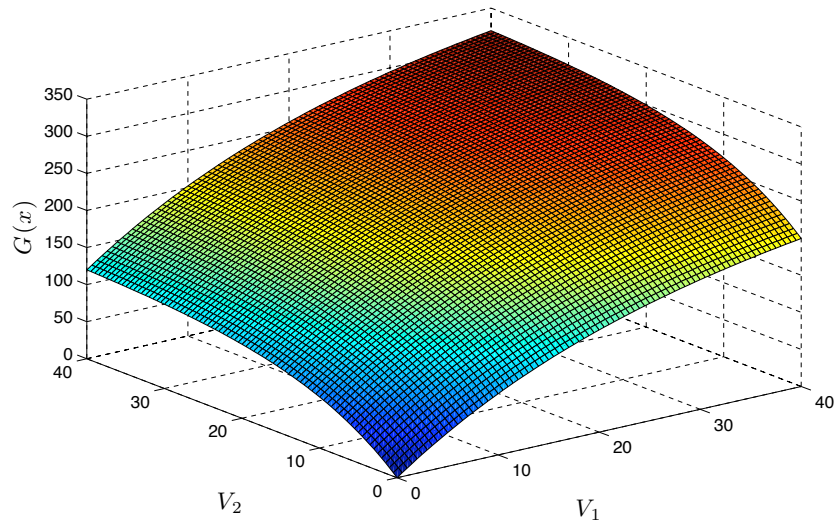
Let us now turn towards permanent price impacts. The academic literature identified two basic qualitative features of market frictions (see, e.g., Almgren and Chriss (2001)): (1) the size of an order influences the transaction price itself and (2) the price of transactions after a large order is affected as well, but to a smaller extent. The former effect is called *temporary* price impact and the latter is called *permanent* price impact. In our original framework we have only dealt with temporary price impacts. Clearly, permanent price impacts play an important role as soon as one considers multiple time periods, such as the problem of optimally liquidating a portfolio over time. As our formalism is essentially static, it is not obvious that we miss out by not considering permanent price impacts. However, a careful study of the optimal liquidation problem and our subsequent use of the optimal liquidity costs shows that we do consider implicitly a period beyond the liquidation as we value the asset portfolio at the frictionless value is unaffected by our liquidation.

Let us define the value of a portfolio $p \in \mathcal{P}$ after we liquidate $0 \leq x \leq p$ without permanent price impacts as follows:

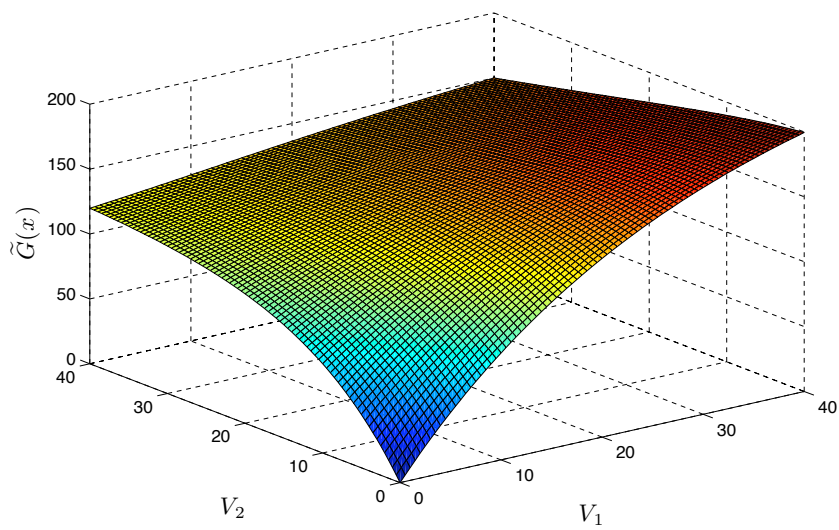
$$V(p, x) := V(p - x) + G(x).$$

Note that we have that $V^\alpha(p) = V(p, x_\alpha^*)$, where $x_\alpha^* := \arg \min \{C(x) \mid G_T(x) = \alpha, 0 \leq x \leq p\}$. Introducing permanent price impacts, we get

$$\tilde{V}(p, x) := V(p - x) - \text{Perma}(p, x) + G(x).$$



(a) Proceeds without asset cross-effect



(b) Proceeds adjusted for asset cross-effects

Figure 4.1: Plots, based on Example 4.1, of the proceeds with (bottom) and without (top) asset cross-effects. The negative effect of cross-effects is clearly visible for liquidations involving large amounts of both assets. Notice that the proceeds with asset cross-effects is not monotone anymore.

We do not go into the desirable properties of Perma here but note that Huberman and Stanzl (2004) argues that permanent price impacts should be linear in transaction sizes to rule out quasi-arbitrage opportunities. The simplest way to incorporate permanent price impacts into our formalism is solely for valuation purposes, leaving the optimization problem as before. In particular, the liquidity-adjusted value of a portfolio given a liquidity call and subject to permanent price impacts is then given by

$$\tilde{V}^\alpha(p) := \begin{cases} \tilde{V}(p, x_a^*), & \text{for } p \in \mathcal{L}^\alpha \\ 0, & \text{for } p \notin \mathcal{L}^\alpha. \end{cases}$$

However, this approach is not as interesting as taking the permanent price impacts into account in the optimization problem. Such a new optimization problem should balance the possible trade-off between temporary and permanent price impacts. We can imagine that a particular asset has low temporary price impacts but large permanent effects. Hence, liquidating this asset is good for servicing the liquidity call but is bad for the ex-post portfolio valuation. First, recall the original optimal liquidation problem:

$$C^\alpha(p) := \begin{cases} \min\{C(x) \mid G(x) = \alpha \text{ for some } 0 \leq x \leq p\}, & \text{for } p \in \mathcal{L}^\alpha \\ V(p), & \text{for } p \notin \mathcal{L}^\alpha. \end{cases}$$

With permanent price impacts, the bank now attempts to minimize the total liquidity costs of meeting the liquidity call, consisting of the immediate liquidity costs and the ex-post valuation effect or maximizing the liquidity-adjusted value:

$$\bar{V}^\alpha(p) := \begin{cases} \sup\{\tilde{V}(p, x) \mid G(x) = \alpha \text{ for some } 0 \leq x \leq p\}, & \text{for } p \in \mathcal{L}^\alpha \\ 0, & \text{for } p \notin \mathcal{L}^\alpha. \end{cases}$$

While conceptually interesting, we believe these two extensions may remain a theoretical exercise as it will be difficult to use them in practice due to data problems.

4.2 Capital allocation revisited

We have seen in Chapter 2 that, if we take a portfolio optimization perspective, Tasche's soundness is a reasonable requirement and leads to the Euler allocation scheme, even in the face of liquidity risk. In this section, we take a different perspective and consider the problem of adjusting the stand-alone risk of a business unit for liquidity risk. In practice, the stand-alone risk contributions play an important role in assessing the risk of a business unit. Formally, the stand-alone risk contribution for monetary risk measures is defined, in the case of no liquidity risk, by

$$\rho_{\text{Sta}}(p_i) := p_i \rho(X_i).$$

The liquidity-adjusted stand-alone risk contribution is ambiguous since we have a portfolio-wide liquidity costs term $C_T^A(\hat{p})$. In order to change this we need to consider

the problem of allocating the optimal liquidity costs per scenario to assets/business units p_1, \dots, p_N .¹ In our model the liquidity call α occurs on a portfolio level, turning the optimal liquidity costs $C^\alpha(p)$ essentially into what is known as *common costs*. There is an extensive literature about the fair allocation of common costs (see, for instance, Hougaard (2009)). It seems reasonable that a “fair” liquidity cost allocation principle should relatively reward the business units that provide liquidity (*liquidity provider*) and relatively penalize the business units that consume liquidity (*liquidity consumer*).

Let us denote by $\Pi_i(p, C^\alpha)$ the allocated optimal liquidity costs of portfolio p and liquidity call α to the i th business unit. Furthermore, let $u_i := \frac{p_i V_i}{V(p)} \in [0, 1]$ denote the portfolio weight of asset i with respect to the MtM value. Clearly, we have that $\sum_{i=0}^N u_i = 1$. Let $x_i^{p,\alpha}$ stand for the i th position of the optimal liquidation strategy of a given $p \in \mathcal{P}$ and liquidity call α . A simple way to capture the qualitative *fairness* property described above is to call Π fair if

$$\text{for } \alpha > 0 \text{ and } x_i^{p,\alpha} > 0 \quad G_i(x_i^{p,\alpha}) \begin{cases} \geq \\ \equiv \\ < \end{cases} \alpha u_i \implies \Pi_i(p, C^\alpha) \begin{cases} \leq \\ \equiv \\ > \end{cases} u_i C^\alpha(p). \quad (4.1)$$

The intuition of the fairness condition is reasonably straightforward: the assets that provide more cash under the optimal liquidation strategy in the case of a liquidity call than the proportional MtM value weighted liquidity call should be allocated less liquidity costs than the proportional MtM value weighted total liquidity costs. That way we reward liquidity providers and penalizes liquidity consumers, taking into account size differences. Next, we collect some basic technical properties that an allocation scheme should have, in addition to the fairness principle.

Definition 4.2 Sound liquidity cost allocation principle. A liquidity cost allocation scheme Π is *sound* if it is fair according to Equation 4.1 and satisfies the properties:

1. **No liquidity costs**

$$C^\alpha(p) = 0 \implies \Pi_i(p, C^\alpha) = 0 \quad \text{for } i = 0, \dots, N,$$

2. **Total allocation**

$$\sum_{i=0}^N \Pi_i(p, C^\alpha) = C^\alpha(p),$$

We suggest to use Definition 4.2 as a yardstick by which allocation principles can be evaluated. Let us consider some natural allocation principles.

Example 4.3.

1. the *proportional liquidity cost allocation principle* allocates to asset i :

$$\Pi_i^{\text{Prop}}(p, C^\alpha) := u_i C^\alpha(p) \quad i = 0, \dots, N.$$

2. the *realized liquidity cost allocation principle* allocates to asset i the realized liquidity costs:

$$\Pi_i^{\text{Real}}(p, C^\alpha) := C(x_i^{p,\alpha}) \quad i = 0, \dots, N.$$

¹Since we consider the allocation of the optimal liquidity costs per scenario, we suppress the A and T notation and write $C^\alpha(p)$ to improve the readability.

3. the normalized *Euler liquidity cost allocation principle* allocates to asset i the normalized partial derivative of the optimal liquidity costs with respect to p_i , scaled by the asset position p_i :²

$$\Pi_i^{\text{Euler}}(p, C^\alpha) := \frac{p_i \frac{\partial C^\alpha(p)}{\partial p_i}}{\sum_{i=0}^N p_i \frac{\partial C^\alpha(p)}{\partial p_i}} C^\alpha(p) \quad i = 0, \dots, N.$$

4. the *liquidity consumer/provider liquidity cost allocation principle* allocates to asset i ³

$$\Pi_i^{\text{LCP}}(p, C^\alpha) := \begin{cases} 0, & \text{if } \alpha = 0, \\ C^\alpha(p) \left(2u_i - \frac{G_i(x_i^{p,\alpha})}{\alpha} \right), & \text{if } 0 < \alpha \leq G(p), \\ C^\alpha(p) \left(2u_i - \frac{G_i(p_i)}{\alpha} \right), & \text{if } \alpha > G(p) \end{cases} \quad i = 0, \dots, N.$$

■

Lemma 4.4.

1. The *proportional liquidity cost allocation principle* Π^{Prop} is not sound.
2. The *realized liquidity cost allocation principle* Π^{Real} is not sound.
3. The *normalized Euler liquidity cost allocation principle* Π^{Euler} is not sound.
4. The *liquidity consumer liquidity cost allocation principle* Π^{LCP} is sound.

Proof of Lemma 4.4.

1. It is easily verified by counter-example that Π^{Real} fails the fairness axiom.
2. It is easily verified by counter-example that Π^{Real} fails the fairness axiom. In fact, the principle works exactly in the other direction as it rewards the illiquid assets and punishes the liquid ones.
3. Let us prove the claim by counterexample. Given a liquidity call function formalization as in Section 2.8. Then for $C^\alpha(p)$ differential in the neighborhood of p_i and excluding the cases that $x_i = 0$ and $x_i = p_i$, the partial derivative scaled by the i th position is given by

$$p_i \frac{\partial C^\alpha(p)}{\partial p_i} = p_i \frac{V_i}{V(p)} \alpha \lambda_p = u_i \alpha \lambda_p$$

The normalized Euler allocation to the i th position is then given by

$$\Pi_i^{\text{Euler}}(p, C^\alpha) = \frac{u_i \alpha \lambda_p}{\alpha \lambda_p \sum_{i=0}^N u_i} C^\alpha(p) = u_i C^\alpha(p),$$

which is the same as the proportional rule and hence fails the fairness axiom.

4. As the case that $\alpha > 0$ and $x_i^{p,\alpha} > 0$ corresponds to $0 < \alpha \leq G(p)$, we need to show that $\Pi_i^{\text{LCP}}(p, C^\alpha) = C^\alpha(p) \left(2u_i - \frac{G_i(x_i^p)}{\alpha} \right)$ satisfies the axiom of Rewards liquidity

²We consider the normalized version to guarantee the total allocation property for the liquidity costs.

³Note that in the worst case an asset gets assigned $2u_i C^\alpha(p)$ using Π^{LCP} and that in a very good case the costs can be negative.

providers. We have that

$$\frac{\Pi_i^{\text{LCP}}(p, C^\alpha)}{C^\alpha(p)} = u_i - \left(\frac{G_i(x_i^{p,\alpha})}{\alpha} - u_i \right) < u_i$$

whenever $G_i(x_i^{p,\alpha}) > \alpha u_i$, which is exactly condition in the axiom. □

Let us illustrate the fairness property in a simple example.

Example 4.5 Realized versus liquidity consumer allocation principle. Consider an asset portfolio $p = (p_0, p_1, p_2) = (0, 10, 10)$, a liquidity call of $\alpha = 100$, MtM values of $V_1 = 10$ and $V_2 = 8$, and exponential asset proceed functions with $\theta_1 = 0.08$ and $\theta_2 = 0.02$. The portfolio-wide liquidity costs are $C^{100}(p) = 11.05$. The optimal liquidation strategy is $x = (0, 3.1, 10)$. Now let us compare the realized and the fair liquidity cost allocation principle. First the realized cost approach:

$$\begin{aligned} \Pi_1^{\text{Real}}(p, C^\alpha) &= 3.1 \times 10 - G_1(3.1) & \Pi_2^{\text{Real}}(p, C^\alpha) &= 10 \times 10 - G_2(10) \\ &= 3.56 & &= 7.49 \end{aligned}$$

The alternative approach gives us:

$$\begin{aligned} \Pi_1^{\text{LCP}}(p, C^\alpha) &= 11.05 \left(2 \frac{100}{180} - \frac{27.49}{100} \right) & \Pi_2^{\text{LCP}}(p, C^\alpha) &= 11.05 \left(2 \frac{80}{180} - \frac{72.51}{100} \right) \\ &= 9.24 & &= 1.81 \end{aligned}$$

Using Π^{Real} does not capture the notion of assets being liquidity provider or liquidity consumer relative to the average portfolio-wide liquidity costs. As a result, Asset 2 gets penalized more so than Asset 1 (3.56 versus 7.49) even though the latter benefits from the existence of the former. This changes when we use Π^{LCP} instead. The cost term penalizes Asset 1 and rewards Asset 2 (9.24 versus 1.81) for the simple reason that Asset 2 is used more in the optimal liquidation strategy (3.10 versus 10) and hence can be seen as a liquidity provider in times of crisis, while Asset 1 is used but to a lesser degree (less than its proportional MtM value) and hence can be seen as a liquidity consumer. ■

Using $\Pi^{\text{LCP}}(p, C^\alpha)$ as our liquidity cost allocation principle, we can define the liquidity-adjusted stand-alone risk contribution as follows:

$$\rho_{\text{Sta}}^A(p_i) := \rho(p_i X_i - \Pi_i^{\text{LCP}}(p, C^\alpha)). \quad (4.2)$$

It should be clear that summing up the liquidity-adjusted stand-alone risk contributions generally does not lead to the overall L-EC, i.e., the total allocation property does not hold. While this is not a problem, since we wanted the stand-alone risk contribution but adjusted for liquidity risk, we can use the liquidity cost allocation principle as a normalizing factor to ensure the total allocation property. We present this idea next.

4.2.1 Alternative risk allocation

Let us define the diversification effect of a business unit with respect to the Euler risk contributions without liquidity risk by

$$\text{Div}(p_i|p) := \rho_{\text{Sta}}(p_i) - \rho_{\text{Euler}}(p_i|p).$$

We know that $\text{Div}(p_i|p) \geq 0$ for homogeneous and subadditive risk measure (Tasche, 2000, 2008). In other words, if risk contributions to a homogeneous and subadditive risk measure are calculated as Euler contributions, then the contributions of single assets will never exceed the assets' stand-alone risks, hence $\text{Div}(p_i|p) \geq 0$.

While formally simply a tautology, it is useful to consider the Euler risk contribution as the sum of the stand-alone risk contribution and the diversification effect, written as follows,

$$\rho_{\text{Euler}}(p_i|p) = \rho_{\text{Sta}}(p_i) - \text{Div}(p_i|p).$$

For practical reasons it would be useful if we could add a liquidity risk effect term to the equation to arrive at the liquidity-adjusted risk contribution of a business unit. Define the total increase in risk due to liquidity risk by

$$\Delta_{\text{Liq}}(\hat{p}) := \rho(X^A(\hat{p})) - \rho(X(p)) = \text{L-EC}(\hat{p}) - \text{EC}(p).$$

A simple candidate for the liquidity-adjusted risk contribution of a business unit i and hence the liquidity risk term is

$$\begin{aligned} \text{L-EC}(p_i|\hat{p}) &:= \rho_{\text{Euler}}(p_i|p) + \frac{\rho(\Pi_i^{\text{LCP}}(\hat{p}))}{\sum_{i=0}^N \rho(\Pi_i^{\text{LCP}}(\hat{p}))} \Delta_{\text{Liq}}(\hat{p}) \\ &= \rho_{\text{Sta}}(p_i) - \text{Div}(p_i|p) + \frac{\rho(\Pi_i^{\text{LCP}}(\hat{p}))}{\sum_{i=0}^N \rho(\Pi_i^{\text{LCP}}(\hat{p}))} \Delta_{\text{Liq}}(\hat{p}) \\ &= \rho_{\text{Sta}}(p_i) - \text{Div}(p_i|p) + \text{Liq}(p_i|\hat{p}), \end{aligned} \quad (4.3)$$

where

$$\text{Liq}(p_i|\hat{p}) := \frac{\rho(\Pi_i^{\text{LCP}}(\hat{p}))}{\sum_{i=0}^N \rho(\Pi_i^{\text{LCP}}(\hat{p}))} \Delta_{\text{Liq}}(\hat{p}).$$

Clearly, the total allocation property holds:

$$\text{L-EC}(\hat{p}) = \sum_{i=0}^N \text{L-EC}(p_i|\hat{p}) = \sum_{i=0}^N \rho_{\text{Sta}}(p_i) - \sum_{i=0}^N \text{Div}(p_i|p) + \sum_{i=0}^N \text{Liq}(p_i|\hat{p}).$$

From a performance measure perspective we can use the following definition of the liquidity-adjusted RAROC of business unit i ,

$$\overline{\text{L-RAROC}}(p_i|\hat{p}) := \frac{p_i E[X_i] - E[\Pi_i(p, C_T^A(\hat{p}))]}{\rho_{\text{Sta}}(p_i) - \text{Div}(p_i|p) + \text{Liq}(p_i|\hat{p})}.$$

4.3 Portfolio dynamics

In our liquidity risk formalism we adopt the standard one period setup of the capital determination problem as proposed by Artzner et al. (1999). In the discussion of

Chapter 2 we recognized that the static setting is a limitation of our formalism. In particular, we agreed that it is difficult to reduce a bank's funding risk to a single liquidity call number, as banks face a dynamic multi-period net cash flow balance that is a complex combination of cash in- and outflows streams arising from their asset and liability portfolio. In addition, recovery actions of banks facing serious liquidity problems are more complex and might involve actions of different durations. While we have chosen simplicity over completeness in this thesis, developing a dynamic multi-period formalism without succumbing to intractability is an interesting topic. Mathematically speaking, modeling all multi-period (stochastic) cash in- and cash outflows of a bank's portfolio pair as well as specify multi-period recovery strategies, would require a multi-period stochastic optimization framework with a multi-variate stochastic process and perhaps a time-consistent dynamic risk measures (see, for instance, Artzner et al. (2007) and Roorda et al. (2005)).

A key difference between the static approach and any dynamic extensions of our formalism is that portfolio positions become stochastic. This occurs even if we disregard exogenous / autonomous portfolio changes,⁴ because meeting liquidity calls leads to portfolio changes. In Chapter 2 we could avoid this aspect because we did not consider periods beyond T . Formally, this means that we move towards random portfolio pairs, i.e., $\hat{p}(\omega)$. Note that this also requires the specification of the relationship between liquidity calls and liability positions, which goes beyond the liquidity call function A function.

Moving to a multi-period setting naturally leads to the notion of anticipating and preparing what may happen in future periods, i.e., optimization. In our context this means that a bank has to balance the minimization of liquidity costs now and being prepared for liquidity calls in future periods. While a multi-period portfolio optimization problem with liquidity risk is interesting in its own rights, it deviates from or at least generalizes the typical capital determination problem that we focused on in this thesis. The line between quasi-autonomous descriptive bank behavior and optimization strategies can become blurry pretty quickly. Either way, our results on the static formalism could be used as a building block for multi-period formalism. We think that stochastic programming is a suitable general framework for extending our formalism to the multi-period setting without losing tractability. A good example in a similar context is the model presented in Jobst et al. (2006).

Another way to extend our formalism to the dynamic setting is to incorporate our ideas into structural Merton-type credit risk models. A common approach to model solvency risk is by *threshold models* based on the firm-value interpretation of default. In this approach the asset-value of a firm is modeled as a nonnegative stochastic process and the value of liabilities are represented by some threshold. In line with economic

⁴Here we mean, for instance, reactions of the bank and bank's investors to scenarios with regard to replacing maturing assets and liabilities, portfolio growths, reductions of exposures, reclassifications of assets as seen during the Subprime crisis etc. These quasi-autonomous descriptive strategies of a bank are sometimes known as management interventions and factored into the market risk and interest rate risk analyses in banks.

intuition, it is assumed that default occurs if the asset value of the firm crosses the value of the liabilities. In the simplest case, one considers a fixed risk management horizon T at which a bank can default. So-called first-passage time models (Black and Cox (1976), Longstaff and Schwartz (1995)) abandon the assumption that insolvency can only occur at some risk management horizon T and assume that default can occur as soon as the threshold is hit. Credit portfolio models used in the industry, such as the popular KMV model, typically use multivariate normal distributions with factor structure for the critical value (so called Gauss-copula models).⁵ In order to incorporate our idea of liquidity risk into the threshold model class is to replace the standard value with our liquidity-adjusted value. This involves two basic assumptions. First, the bank is subject to random liquidity shocks. Such shocks reflect unexpected liquidity cash calls. With a given probability the bank has to sell part of its position immediately to generate the cash call. The realized optimal liquidity costs decrease the bank's asset value on top of any non-liquidity risk decreases. More formally, let us denote the value of the bank's assets at time $t > 0$ by $V_t(p_t)$. It is the bank's value just before it learns what the size of the liquidity call is given by

$$V_t(p) = \sum_{i=0}^N p_{t,i} V_{t,i}.$$

As discussed before we need to keep track of portfolio changes. Conditional on the realization of a liquidity call of size α_t the liquidity-adjusted value of the bank at time t is given by

$$V_t^\alpha(p_t) = V_t(p_t) - C_t^\alpha(p_t).$$

In line with first-passage model, default occurs at the stopping time:

$$\tau := \inf\{t \geq 0 \mid V_t^\alpha(p_t) \leq D_t\},$$

where D_t is a, possibly stochastic, default barrier. Notice that via the 100% rule it might be possible that the liquidity-adjusted value jumps to zero despite being far away from the default barrier. Now the default indicator variable for a given risk management horizon T is

$$Y := \mathbf{1}_{\{\tau \leq T\}}$$

and the corresponding PD is given by

$$\text{PD}_{(0,T)} := \mathbb{P}\{Y\}.$$

Applying the above ideas requires the modeling of the liquidity calls as well as the proceeds functions as stochastic processes. The simplest approach is to model the liquidity calls as a top-down stochastic process correlated with the bank's asset value. For instance, if ensure that the correlation is negative, we can model that during times of solvency pressure the likelihood of liquidity shocks would tend to increase and hence

⁵It is interesting that various threshold models, including the KMV model, can be reformulated as a mixture model (see McNeil et al. (2005)), similar to the one we used in Chapter 3 to model the credit risk of the bank's loan book and the bank's liquidity calls.

the default risk. Modeling the proceed function as a stochastic process can become very complex if we do not simplify considerably as we have done throughout the examples in this thesis.

References

- Almgren, R. and Chriss, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–29. 4.1
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent risk measures. *Mathematical Finance*, 9:203–228. 4.3
- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., and Ku, H. (2007). Coherent multiperiod risk adjusted values and bellman's principles. *Annals of Operations Research*, 152:5–22. 4.3
- Black, F. and Cox, J. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31:351–367. 4.3
- Hougaard, J. L. (2009). *An Introduction to Allocation Rules*. Springer. 4.2
- Huberman, G. and Stanzl, W. (2004). Price manipulation and quasi-arbitrage. *Econometrica*, 72:1247–1275. 4.1
- Jobst, N. J., Mitra, G., and Zenios, S. A. (2006). Integrating market and credit risk: A simulation and optimisation perspective. *Journal of Banking & Finance*, 30:717–742. 4.3
- Longstaff, F. A. and Schwartz, E. S. (1995). A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, 50:789–819. 4.3
- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press. 5
- Roorda, B., Schumacher, J., and Engwerda, J. (2005). Coherent acceptability measures in multiperiod models. *Mathematical Finance*, 15:589–612. 4.3
- Schoenborn, T. (2008). *Trade execution in illiquid markets - optimal stochastic control and multi-agent equilibria*. PhD thesis, Technische Universitaet Berlin. 4.1
- Tasche, D. (2000). Risk contributions and performance measurement. www-m4.ma.tum.de/pers/tasche/riskcon.pdf. Working paper, February. 4.2.1
- Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the euler principle. In Resti, A., editor, *Pillar II in the New Basel Accord: The Challenge of Economic Capital*, pages 423–453. Risk Books. 4.2.1

5

Conclusions

It has been the purpose of this thesis to develop a mathematical framework for measuring a bank's liquidity risk and study its properties. In particular, we set out to make economic capital and RAROC sensitive to liquidity risk in a rigorous way and address the problem of capital allocation under liquidity risk.

We introduced the concept of optimal liquidity costs and liquidity cost profiles as a quantification of a bank's illiquidity at balance sheet level. This led to the key concept of liquidity-adjusted risk measures defined on the vector space of asset and liability pairs under liquidity call functions. While the explicit move towards the vector space of balance sheets (portfolio pairs) has not been entirely new, since Acerbi and Scandolo (2008) proposed something similar, we expanded on previous results by introducing liquidity call functions and by maintaining the important role played by Type 2 liquidity risk. We studied the model-free effects of adding, scaling, and mixing balance sheets. In particular, we could show that convexity and positive super-homogeneity of risk measures is preserved in terms of positions under the liquidity adjustment, given certain conditions with regard to Type 2 liquidity risk are met, while coherence is not, reflecting the common idea that size does matter. Nevertheless, we argued that coherence remains a natural assumption at the level of *underlying* risk measures for its reasonable properties in the absence of liquidity risk. Convexity shows that even under liquidity risk the concept of risk diversification survives. Positive super-homogeneity confirms the common intuition that the level of riskiness generally increases with increased position size when liquidity risk is present. We showed that liquidity cost profiles can be used to determine whether combining positions is beneficial or harmful. In particular, we have shown that combining positions with the same marginal liquidity costs generally leads to an increase of total liquidity costs. This effect works in opposite

direction of the subadditivity of the underlying risk measure, showing that a merger can create extra risk in the presence of liquidity risk. Finally, we addressed the liquidity-adjustment of the well-known Euler allocation principle for risk capital. We could show that such an adjustment is possible without losing the soundness property that justifies the Euler principle. However, it is in general not possible to combine soundness with the total allocation property for both the numerator and the denominator in liquidity-adjusted RAROC.

In addition, we have presented an illustration of the formalism in the context of a semi-realistic economic capital setting. We characterized the bank's funding liquidity risk with the help of a Bernoulli mixture model, using the bank's capital losses as the mixing variable, and use standard marginal risk models for credit, market, and operational risk. After deriving the joint model using a copula, we analyzed the impact of balance sheet composition on liquidity risk. Furthermore, we developed a simple, robust, and efficient numerical algorithm based on the results in Lemma 2.10 for the computation of the optimal liquidity costs per scenario. While the optimization problem behind the liquidity cost term is convex and hence readily solvable with standard software tools, our algorithm is generally more efficient. We have shown that even a simple but reasonable implementation of liquidity risk modeling can lead to a significant deterioration of capital requirements and risk-adjusted performance for banks with safe funding but illiquid assets, exemplified by the retail bank, and banks with liquid assets but risky funding, exemplified by the investment bank. In addition, we have shown that the formal results of Theorem 2.26 on p. 38 are relevant, especially the super-homogeneity result of liquidity-adjusted risk measures. Overall bank size and the non-linear scaling effects of liquidity risk become apparent for banks that have to rely on a large amount of fire selling. Overall, our illustration confirms the common intuition that a bank's liquidity risk management must involve the careful balancing between the market liquidity risk of its assets and the funding risk of its liabilities.

5.1 Limitations and future research

While we think that the formalism presented in this thesis is a step in the right direction in making liquidity risk measurement and management more rigorous, there are several limitations as well as promising opportunities for future research.

The concept of random liquidity call functions is crucial in describing the interaction between funding and market liquidity risk in our formalism. While we have presented some ideas in our illustration, they were still highly stylized. Realistic modeling of this concept is a critical, yet underdeveloped research topic. The more we know about what risk factors affect a bank's liquidity risk and about how these factors can be influenced, the more advice we will be able to give banks and policy makers about how to control liquidity risk of individual banks and hence improve financial stability as a whole.

In our analysis, liquidity risk has been inherently static: banks faced a single liq-

liquidity call at some time horizon and had to optimally recover from it instantaneously. Practitioners might argue that our static approach neglects essential dynamic (timing) elements of liquidity risk. In particular, funding risk cannot be reduced to a single liquidity call number, as banks face a dynamic multi-period net cash flow balance that is a complex combination of cash in- and outflows streams arising from its asset and liability portfolio. In addition, recovery actions of banks facing serious liquidity problems are more complex and might involve actions of different durations. The feasibility of these measures are a function of the nature, severity, and duration of the liquidity shocks, i.e., they are a function of the state of the world and time. Developing a dynamic multi-period framework, without succumbing to intractability, is an interesting topic for future research. Apart from realistic but tractable multi-period stochastic models, a generalization of our static formalism to the dynamic setting may require the use of time-consistent risk measures (dynamic risk measure theory) as well as a clear formalization of the space acceptable strategies.

Finally, we think that addressing the link between a bank's interest rate risk and our formalism deserves some attention. On the one side, there is the connection between current ALM techniques in banks and the liquidity call function that we have not covered in detail in this thesis. On the other side, there is the link between the allocation of liquidity costs as presented in our formalism and funds transfer pricing frameworks used in practice by banks.

5.2 Implications

Our results may have implications for financial regulations and banks. Liquidity-adjusted risk measures could be a useful addition to banking regulation and bank management, as they capture essential features of a bank's liquidity risk, can be combined with existing risk management systems, possess reasonable properties under portfolio manipulations, and lead to an intuitive risk ranking of banks. In fact, our framework may be seen as a more elaborate and rigorous version of the Liquidity Coverage Ratio of Basel III (BIS, 2010). Furthermore, combining our framework with the ideas of mark-to-funding in Brunnermeier et al. (2009) and "CoVaR" in Adrian and Brunnermeier (2009) may help regulators manage systemic risk, originating from bank's individual liquidity risk exposure. Internally, banks could use liquidity-adjusted Economic Capital and liquidity-adjusted RAROC, as well as the allocation schemes, to manage their risk-reward profile.

Our findings regarding the properties of liquidity-adjusted risk measures may have implications for banks and risk measure theory. The result that under liquidity risk the diversification effect of risk is not generally ensured despite using a subadditive underlying risk measure and the result that size matters may affect how bank managers and financial regulators think about bank expansions and mergers. Furthermore, our findings regarding the preservation of convexity under liquidity risk strengthens the argument for using coherent risk measures as underlying risk measures due to their

reasonable properties in the absence of liquidity risk.

References

- Acerbi, C. and Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, 8:681–692. 5
- Adrian, T. and Brunnermeier, M. K. (2009). Covar. <http://ssrn.com/abstract=1269446>. Working paper, Federal Reserve Bank of New York / Princeton University, May. 5.2
- BIS (2010). Basel 3: International framework for liquidity risk measurement, standards and monitoring. Bank of International Settlements. 5.2
- Brunnermeier, M., Crocket, A., Goodhart, C., Persaud, A., and Shin, H. S. (2009). The fundamental principles of financial regulation. *Geneva Reports on the World Economy*, 11. 5.2

Bibliography

Chapter 1

- Acerbi, C. and Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, 8:681–692. 1.3
- Anderson, W., Liese, M., and Weber, S. (2010). Liquidity risk measures. http://www.ims.nus.edu.sg/Programs/financialm09/files/weber_tut1b.pdf. Working paper. 1.3
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent risk measures. *Mathematical Finance*, 9:203–228. 1.1, 4, 1.2, 1.3
- BIS (2000). Sound practices for managing liquidity in banking organisations. Bank of International Settlements. 1.2
- BIS (2006). The management of liquidity risk in financial groups. Bank of International Settlements. 1.2
- BIS (2008a). Liquidity risk: Management and supervisory challenges. Bank of International Settlements. 1.2
- BIS (2008b). Principles for sound liquidity risk management and supervision. Bank of International Settlements. 1.2
- BIS (2009). Range of practices and issues in economic capital frameworks. Bank of International Settlements. 8
- BIS (2010). Basel 3: International framework for liquidity risk measurement, standards and monitoring. Bank of International Settlements. 1.1, 1.2
- Brunnermeier, M., Crocket, A., Goodhart, C., Persaud, A., and Shin, H. S. (2009). The fundamental principles of financial regulation. *Geneva Reports on the World Economy*, 11. 1.1
- Buehlmann, H. (1997). The actuary: the role and limitations of the profession since the mid-19th century. *ASTIN Bulletin*, 27:165–171. 1.1
- Citigroup Inc. (2008). Annual report. <http://www.citibank.com/>. 1.2
- Deutsche Bank (2008). Annual report: Risk report. <http://www.db.com/>. 1.2
- Jarrow, R. A. and Protter, P. (2005). Liquidity risk and risk measure computation. *Review of Futures Markets*, 11:27–39. 1.3
- JPMorgan Chase & Co. (2007). Form 10-k. <http://www.jpmorgan.com/>. 1.2
- Klaassen, P. and van Eeghen, I. (2009). *Economic Capital - How it works and what every manager should know*. Elsevier. 1.1
- Ku, H. (2006). Liquidity risk with coherent risk measures. *Applied Mathematical Finance*, 13:131–141. 1.3
- Lannoo, K. and Casey, J.-P. (2005). Capital adequacy vs. liquidity requirements in banking supervision in the eu. Technical report, Centre for European Policy Studies. 9
- Martin, A. (2007). Liquidity stress testing - scenario modelling in a globally operating bank. In *APRA Liquidity Risk Management Conference*. 1.2
- Matz, L. and Neu, P. (2007). *Liquidity Risk Measurement and Management - A practitioner's guide to global best practices*. Wiley. 1.2
- Morris, S. and Shin, H. S. (2009). Illiquidity component of credit risks. <http://www.princeton.edu/~hsshin/www/IlliquidityComponent.pdf>. Working paper, Princeton University, September. 1.1
- Nikolaou, K. (2009). Liquidity (risk) concepts - definitions and interactions. <http://www.ecb.int/pub/pdf/scpwps/ecbwp1008.pdf>. Bank of England Working Paper, No 1008, February. 1.2
- Standard&Poor's (2004). Fi criteria: Rating banks. <http://www.standardandpoors.com/>. 1.2
- The Goldman Sachs Group (2006). Form 10-k. <http://www2.goldmansachs.com>. 1.2
- UBS AG (2008). Annual report: Risk and treasury management. <http://www.ubs.com/>. 1.2

Chapter 2

- Acerbi, C. (2004). Coherent representations of subjective risk-aversion. In Szegoe, G., editor, *Risk Measure for the 21st Century*, pages 147–207. Wiley Finance. 2.4.4, 2.5.3
- Acerbi, C. and Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, 8:681–692. 2.1, 2.2, 2.2, 8, 2.2, 19, 2.5.3, 2.5.4, 2.5.4
- Adrian, T. and Brunnermeier, M. K. (2009). Covar. <http://ssrn.com/abstract=1269446>. Working paper, Federal Reserve Bank of New York / Princeton University, May. 2.10
- Alfonsi, A., Fruth, A., and Schied, A. (2010). Optimal execution strategies in limit order books with general shape functions. *Quantitative Finance*, 10:143–157. 2.2
- Allen, F. and Carletti, E. (2008). The role of liquidity in financial crises. <http://ssrn.com/abstract=1268367>. Working paper, September. 2.2.2
- Almgren, R. (2003). Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance*, 10:1–18. 2.1, 2.2
- Almgren, R. and Chriss, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–29. 2.1, 2.2
- Anderson, W., Liese, M., and Weber, S. (2010). Liquidity risk measures. http://www.ims.nus.edu.sg/Programs/financialm09/files/weber_tut1b.pdf. Working paper. 2.1, 2.2, 2.2.3, 2.4.4, 20, 2.4.4, 2.5.1
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent risk measures. *Mathematical Finance*, 9:203–228. 2.1, 2.4.2, 2.5.3, 26
- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., and Ku, H. (2007). Coherent multiperiod risk adjusted values and bellman's principles. *Annals of Operations Research*, 152:5–22. 2.9.1
- Berkowitz, J. (2000). A coherent framework for stress-testing. *Journal of Risk*, 2:5–15. 2.7
- BIS (2005). International convergence of capital measurement and capital standards: A revised framework. Bank of International Settlements. 2.7
- BIS (2009). Range of practices and issues in economic capital frameworks. Bank of International Settlements. 2.4.4, 2.5.3
- BIS (2010). Basel 3: International framework for liquidity risk measurement, standards and monitoring. Bank of International Settlements. 2.10
- Bouchaud, J. (2009). Price impact. <http://arxiv.org/abs/0903.2428>. Entry for the upcoming Encyclopedia of Quantitative Finance. 2.2
- Breuer, T., Jandacka, M., Rheinberger, K., and Summer, M. (2009). How to find plausible, severe and useful stress scenarios. *International Journal of Central Banking*, 5:206–224. 2.7
- Brigo, D. and Nardio, C. (2010). Liquidity-adjusted market risk measures with stochastic holding period. Working paper, King's College London, October. 2.2.1, 2.7
- Brunnermeier, M., Crocket, A., Goodhart, C., Persaud, A., and Shin, H. S. (2009). The fundamental principles of financial regulation. *Geneva Reports on the World Economy*, 11. 2.4.4, 2.10
- Danielsson, J., Jorgensen, B. N., Sarma, M., and Vries, C. G. D. (2005). Sub-additivity re-examined: The case for value-at-risk. Working paper, Eurandom. 28
- Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, 4:1–34. 2.6.1, 2.6.2
- Duffie, D. and Ziegler, A. (2003). Liquidation risk. *Financial Analysts Journal*, 42. 2.3.1
- Engle, R. and Ferstenberg, R. (2007). Execution costs. *Journal of Portfolio Management*, 33:34–45. 2.1
- Foellmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, 6:429–447. 2.5.3
- Frittelli, M. and Gianin, E. (2002). Putting order in risk measures. *Journal of Banking and Finance*, 26:1473–1486. 2.5.3
- Gibson, M. S. (2001). Incorporating event risk into value-at-risk. Federal Reserve Board. 2.7
- Glosten, L. and Milgrom, P. (1985). Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics*, 14:71–100. 2.2
- Gruendl, H. and Schmeiser, H. (2007). Capital allocation for insurance companies: What good is it? *Journal of Risk and Insurance*, 74:301–317. 2.6.2
- Hallerbach, W. G. (2004). Capital allocation, portfolio enhancement and performance measurement: a unified approach. In Szegoe, G. P., editor, *Risk Measures for the 21st Century*, pages 435–450. John Wiley & Sons. 2.6.2
- Heath, D. and Ku, H. (2004). Pareto equilibria with coherent measures of risk. *Mathematical Finance*,

- 14:163–172. 2.5.3
- Jarrow, R. A. and Protter, P. (2005). Liquidity risk and risk measure computation. *Review of Futures Markets*, 11:27–39. 2.1, 2.2, 2.2.1, 2.5.3
- Jarrow, R. A. and Purnanandam, A. K. (2005). A generalized coherent risk measure from the firm's perspective. *Finance Research Letters*, 2:23–29. 2.4.1, 2.4.4
- Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance*, 15:425–437. 2.6.1
- Klaassen, P. and van Eeghen, I. (2009). *Economic Capital - How it works and what every manager should know*. Elsevier. 2.4.3, 2.4.4, 2.5.3
- Krokhmal, P. and Uryasev, S. (2007). A sample-path approach to optimal position liquidation. *Annals of Operations Research*, 152:193–225. 2.1
- Ku, H. (2006). Liquidity risk with coherent risk measures. *Applied Mathematical Finance*, 13:131–141. 2.1
- Kyle, A. S. (1985). Continuous auctions and insider trading. *Econometrica*, 53:1315–1336. 2.2
- Matz, L. and Neu, P. (2007). *Liquidity Risk Measurement and Management - A practitioner's guide to global best practices*. Wiley. 4, 21
- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press. 1, 2.2.1, 24
- Pflug, G. C. and Roemisch, W. (2007). *Modeling, Measuring, and Managing Risk*. World Scientific Publishing. 2.6.2
- Plantin, G., Shin, H. S., and Sapra, H. (2008). Marking to market: Panacea or Pandora's box? *Journal of Accounting Research*, 46:435–460. 2.2
- Powers, M. R. (2007). Using aumann-shapley values to allocate insurance risk: The case of inhomogeneous losses. *North American Actuarial Journal*, 11:113–127. 2.6.2
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. (2007). *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press. 2.10
- Quarteroni, A., Sacco, R., and Saleri, F. (2007). *Numerical mathematics*. Springer. 2.10
- Rockafellar, R. T. and Wets, R. J.-B. (2004). *Variational Analysis*. Springer-Verlag. 2.5.2
- Roorda, B., Schumacher, J., and Engwerda, J. (2005). Coherent acceptability measures in multiperiod models. *Mathematical Finance*, 15:589–612. 2.9.1
- Schied, A. and Schoenborn, T. (2009). Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13:181–204. 2.1
- Schoenborn, T. (2008). *Trade execution in illiquid markets - optimal stochastic control and multi-agent equilibria*. PhD thesis, Technische Universität Berlin. 2.2
- Subramanian, A. and Jarrow, R. A. (2001). The liquidity discount. *Mathematical Finance*, 11:447–474. 2.1
- Tasche, D. (2000). Risk contributions and performance measurement. www-m4.ma.tum.de/pers/tasche/riskcon.pdf. Working paper, February. 2.6.1, 2.6.2
- Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the Euler principle. In Resti, A., editor, *Pillar II in the New Basel Accord: The Challenge of Economic Capital*, pages 423–453. Risk Books. 2.6.1, 2.6.2, 2.8, 2.10

Chapter 3

- BIS (2009). Range of practices and issues in economic capital frameworks. Bank of International Settlements. 18
- Botev, Z. I., Grotowski, J. F., and Kroese, D. P. (2010). Kernel density estimation via diffusion. *Annals of Statistics*, 38:2916–2957. 3.8
- Demarta, S. and McNeil, A. (2005). The t copula and related copulas. *International Statistical Review*, 73:111–130. 3.6.2
- Feller, W. (1968). *An Introduction to Probability Theory and Its Applications (Volume 1)*. Wiley. 3.8
- González, F., Cossin, D., Huang, Z., and Aunon-Nerin, D. (2003). A framework for collateral risk control determination. <http://ssrn.com/abstract=376233>. Working paper ECB, January. 3.4.1
- Hull, J. C. (2010). *Risk Management and Financial Institutions*. Pearson Education. 15
- Matz, L. and Neu, P. (2007). *Liquidity Risk Measurement and Management - A practitioner's guide to global best practices*. Wiley. 3.5

- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press. 3.4.2, 3.1, 3.4.2, 3.4.2, 3.2, 3.6.2, 13, 3.8, 3.8
- Rosenberg, J. V. and Schuermann, T. (2006). A general approach to integrated risk management with skewed, fat-tailed risks. *Journal of Financial Economics*, 79:569–614. 3.4.4, 3.3, 3.6.2
- Tasche, D. (2000). Risk contributions and performance measurement. www-m4.ma.tum.de/pers/tasche/riskcon.pdf. Working paper, February. 3.8
- Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the euler principle. In Resti, A., editor, *Pillar II in the New Basel Accord: The Challenge of Economic Capital*, pages 423–453. Risk Books. 3.8
- Vasicek, O. (1977). An equilibrium characterisation of the term structure. *Journal of Financial Economics*, 5:177–188. 3.4.1

Chapter 4

- Almgren, R. and Chriss, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–29. 4.1
- Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent risk measures. *Mathematical Finance*, 9:203–228. 4.3
- Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., and Ku, H. (2007). Coherent multiperiod risk adjusted values and bellman's principles. *Annals of Operations Research*, 152:5–22. 4.3
- Black, F. and Cox, J. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31:351–367. 4.3
- Hougaard, J. L. (2009). *An Introduction to Allocation Rules*. Springer. 4.2
- Huberman, G. and Stanzl, W. (2004). Price manipulation and quasi-arbitrage. *Econometrica*, 72:1247–1275. 4.1
- Jobst, N. J., Mitra, G., and Zenios, S. A. (2006). Integrating market and credit risk: A simulation and optimisation perspective. *Journal of Banking & Finance*, 30:717–742. 4.3
- Longstaff, F. A. and Schwartz, E. S. (1995). A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, 50:789–819. 4.3
- McNeil, A., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press. 5
- Roorda, B., Schumacher, J., and Engwerda, J. (2005). Coherent acceptability measures in multiperiod models. *Mathematical Finance*, 15:589–612. 4.3
- Schoenborn, T. (2008). *Trade execution in illiquid markets - optimal stochastic control and multi-agent equilibria*. PhD thesis, Technische Universität Berlin. 4.1
- Tasche, D. (2000). Risk contributions and performance measurement. www-m4.ma.tum.de/pers/tasche/riskcon.pdf. Working paper, February. 4.2.1
- Tasche, D. (2008). Capital allocation to business units and sub-portfolios: the euler principle. In Resti, A., editor, *Pillar II in the New Basel Accord: The Challenge of Economic Capital*, pages 423–453. Risk Books. 4.2.1

Chapter 5

- Acerbi, C. and Scandolo, G. (2008). Liquidity risk theory and coherent measures of risk. *Quantitative Finance*, 8:681–692. 5
- Adrian, T. and Brunnermeier, M. K. (2009). Covar. <http://ssrn.com/abstract=1269446>. Working paper, Federal Reserve Bank of New York / Princeton University, May. 5.2
- BIS (2010). Basel 3: International framework for liquidity risk measurement, standards and monitoring. Bank of International Settlements. 5.2
- Brunnermeier, M., Crocket, A., Goodhart, C., Persaud, A., and Shin, H. S. (2009). The fundamental principles of financial regulation. *Geneva Reports on the World Economy*, 11. 5.2

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Liquidity risk is a crucial and inherent feature of the business model of banks. While banks and regulators use sophisticated mathematical methods to measure a bank's solvency risk, they use relatively simple tools for a bank's liquidity risk such as coverage ratios, sensitivity analyses, and scenario analyses. In this thesis we present a more rigorous framework that allows us to measure a bank's liquidity risk within the standard economic capital and RAROC setting. In particular, we introduce the concept of liquidity-adjusted risk measures defined on the vector space of balance sheet positions under liquidity call functions. Liquidity-adjusted risk measures could be a useful addition to banking regulation and bank management as they capture essential features of a bank's liquidity risk, can be combined with existing risk management systems, possess reasonable properties under portfolio manipulations, and lead to an intuitive risk ranking of banks.

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